International Journal of Modern Physics A
Vol. 24, No. 22 (2009) 4157–4167

World Scientific
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OCTONIC FIRST-ORDER EQUATIONS OF RELATIVISTIC QUANTUM MECHANICS

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Received 3 February 2009
Revised 24 March 2009

We demonstrate a generalization of relativistic quantum mechanics using eight-component octonic wave function and octonic spatial operators. It is shown that the second-order equation for octonic wave function describing particles with spin 1/2 can be reformulated in the form of a system of first-order equations for quantum fields, which is analogous to the system of Maxwell equations for the electromagnetic field. It is established that for the special types of wave functions the second-order equation can be reduced to the single first-order equation analogous to the Dirac equation. At the same time it is shown that this first-order equation describes particles, which do not have quantum fields.

Keywords: Octons; Clifford algebra; octonic quantum mechanics; Proca–Maxwell equation; Dirac equation.

PACS numbers: 03.65.-w, 03.65.Fd, 03.65.Pm.

1. Introduction

The relativistic quantum first-order equation has been formulated by P. A. M. Dirac[1,2] in 1928 on the basis of matrix operators and spinor wave function. But as it was shown in recent years the quadratic form of the Einstein relation for energy and momentum can be realized by different algebras. In particular, there are various algebraic approaches to the generalization of the Dirac equation using different systems of multicomponent hypernumbers such as quaternions,[3–6] biquaternions,[7,8] octonions[9–12] and multivectors generating associative Clifford algebras.[13–15] However, attempts to describe relativistic particles by means of hypercomplex wave functions and to develop geometrical interpretation of the Dirac equation have not made appreciable progress. For example, the few attempts to describe relativistic particles by means of octonion wave functions are confronted by difficulties connected with octonions nonassociativity.[10] Moreover, all systems of hypercomplex numbers, which have been applied up to now for the generalization of quantum

Table 1. The rules of multiplication and commutation for the octon's unit vectors.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$	$\mathbf{a_0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$	$\mathbf{a_1}$	$\mathbf{a_0}$	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$	$\mathbf{a_2}$	$-i\mathbf{a_3}$	$\mathbf{a_0}$	$i\mathbf{a_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1	$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	$\mathbf{a_0}$
$\mathbf{a_0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$	1	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{a_1}$	$\mathbf{a_0}$	$i\mathbf{a_3}$	$-i\mathbf{a_2}$	$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	$\mathbf{a_0}$	$i\mathbf{a_1}$	$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	$\mathbf{a_0}$	$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

mechanics are the objects of hypercomplex space and do not have any consistent space-geometric interpretation.

Recently, we proposed eight-component values "octons"[16] generating a closed noncommutative associative algebra and having a clear well-defined geometric interpretation. From the geometrical point of view an octon is the object of the real three-dimensional space. It is the sum of a scalar, pseudoscalar, vector and pseudovector. In Ref. 17 octons were successfully applied to the generalization of relativistic quantum mechanics on the basis of eight-component octonic wave function and octonic spatial operators. It was shown that the octonic second-order equation correctly describes relativistic particles with spin 1/2 in an external electromagnetic field.

In this paper we propose the system of the first-order octonic equations for quantum fields analogous to the Maxwell equations for electromagnetic field and discuss the octonic generalization of the Dirac equation.

2. Octonic Wave Function and Spatial Inversion

We will consider the wave function of a relativistic particle in the form of an eight-component octon[17]

$$\breve{\psi} = \varphi_0 + \varphi_1\mathbf{e_1} + \varphi_2\mathbf{e_2} + \varphi_3\mathbf{e_3} + \chi_0\mathbf{a_0} + \chi_1\mathbf{a_1} + \chi_2\mathbf{a_2} + \chi_3\mathbf{a_3}\,. \tag{1}$$

The components $\varphi_\alpha(\vec{r}, t)$ and $\chi_\alpha(\vec{r}, t)$ $(\alpha = 0, 1, 2, 3)$ are scalar (complex in general) functions of spatial coordinates and time. The values $\mathbf{e}_k$ $(k = 1, 2, 3)$ are axial unit vectors, $\mathbf{a}_k$ are polar unit vectors, $\mathbf{a_0}$ is the pseudoscalar unit. The algebra of octons was discussed in detail in Ref. 16. Here we briefly recall the rules of multiplication and commutation for octonic basis elements, which are represented in Table 1. In Table 1 and below the value i is the imaginary unit $(i^2 = -1)$.

We can indicate some connection between octonic algebra and algebra of quaternions. Indeed, it is easy to see that the rules of octonic multiplication and commutation take place for the values based on the quaternionic imaginary units q_k $(k = 1, 2, 3; q_k^2 = -1)$

$$\mathbf{e}_k = iq_k\,, \qquad \mathbf{a}_k = iq_k\mathbf{a_0}\,,$$

but it needs the introduction of a new (nonquaternionic) pseudoscalar element $\mathbf{a_0}$.

Besides, we can introduce the generators

$$g_l^{\pm} = \frac{1}{2i}\,(a_l \pm e_l) \quad (l = 1, 2, 3)\,, \tag{2}$$

which have the "quaternionic" rules of multiplication and commutation:

$$g_l^{\pm} g_m^{\pm} = \mathcal{E}_{lmn} g_n^{\pm}\,, \qquad g_l^{\pm} g_m^{\mp} = 0 \quad (m, n = 1, 2, 3)\,, \tag{3}$$

where $\mathcal{E}_{lmn}$ is the unit antisymmetric tensor. At that the values $P^{\pm} = \frac{1}{2}(1 \pm a_0)$ are projectors:

$$\begin{aligned} P^{\pm} g_l^{\pm} &= g_l^{\pm} P^{\pm} = g_l^{\pm}\,, \qquad & P^{\pm} g_l^{\mp} &= g_l^{\mp} P^{\pm} = 0\,, \\ g_l^{\pm} g_l^{\pm} &= P^{\pm}\,, & P^{\pm} P^{\pm} &= P^{\pm}\,. \end{aligned} \tag{4}$$

The octonic wave function (1) can also be written in the compact form

$$\breve{\psi} = \varphi_0 + \overleftrightarrow{\varphi} + \tilde{\chi}_0 + \vec{\chi}\,, \tag{5}$$

where the pseudovector part is indicated by a double arrow "$\leftrightarrow$," the pseudoscalar part by a wave "$\sim$" and the vector part by an arrow "$\rightarrow$." Note that unit elements of octonic basis $\mathbf{e}_k$ and $\mathbf{a}_k$ play the role of some octonic operators $\hat{\mathbf{e}}_k$ and $\hat{\mathbf{a}}_k$, which transform the spatial structure of the wave function by means of octonic multiplication. For example, the action of the operator $\hat{\mathbf{a}}_\mathbf{1}$ can be represented as octonic multiplication of the unit vector $\mathbf{a_1}$ and octon $\breve{\psi}$:

$$\hat{\mathbf{a}}_\mathbf{1}\breve{\psi} = \mathbf{a_1}\breve{\psi} = \chi_1 + \chi_0\mathbf{e_1} - i\chi_3\mathbf{e_2} + i\chi_2\mathbf{e_3} + \varphi_1\mathbf{a_0} + \varphi_0\mathbf{a_1} - i\varphi_3\mathbf{a_2} + i\varphi_2\mathbf{a_3}\,. \tag{6}$$

The matrix representation of octonic spatial operators and their eigenfunctions were considered in Ref. 17. Further, we will use symbolic designations $\hat{\mathbf{e}}_k$ and $\hat{\mathbf{a}}_k$ in the operator part of equations but $\mathbf{e}_k$ and $\mathbf{a}_k$ designations in wave functions.

Let us define the operation of local spatial inversion (R) of octonic wave function. This operation reverses the vector component of the wave function and changes the sign of the pseudoscalar component. In particular, the simplest octonic wave functions corresponding to the elements of octon's basis are transformed under spatial inversion in accordance with the following rules:

$$R : \mathbf{a}_k \Rightarrow -\mathbf{a}_k\,, \tag{7}$$

$$R : \mathbf{e}_k \Rightarrow \mathbf{e}_k\,, \tag{8}$$

$$R : \mathbf{a_0} \Rightarrow -\mathbf{a_0}\,. \tag{9}$$

The operation of spatial inversion is realized by operator $\hat{R}$, which changes the signs of vector and pseudoscalar components of the wave function:

$$\hat{R}\big(\varphi_0 + \overleftrightarrow{\varphi} + \tilde{\chi}_0 + \vec{\chi}\big) = \big(\varphi_0 + \overleftrightarrow{\varphi} - \tilde{\chi}_0 - \vec{\chi}\big)\,, \tag{10}$$

$$\hat{R}^2 = 1\,. \tag{11}$$

We specially emphasize that the operator $\hat{R}$ does not transform arguments of the wave function, i.e. for example $\hat{R}\vec{\chi}(x,y,z,t) = -\vec{\chi}(x,y,z,t)$. The important property of operator $\hat{R}$ is the anticommutation with $\hat{\mathbf{a}}_{\mathbf{0}}$, $\hat{\mathbf{a}}_{\mathbf{1}}$, $\hat{\mathbf{a}}_{\mathbf{2}}$, $\hat{\mathbf{a}}_{\mathbf{3}}$ and the commutation with $\hat{\mathbf{e}}_{\mathbf{1}}$, $\hat{\mathbf{e}}_{\mathbf{2}}$, $\hat{\mathbf{e}}_{\mathbf{3}}$ operators. Indeed, it is simple to check directly that

$$\hat{R}\hat{\mathbf{a}}_{\mathbf{0}}\breve{\psi} = -\hat{\mathbf{a}}_{\mathbf{0}}\hat{R}\breve{\psi}\,, \tag{12}$$

$$\hat{R}\hat{\mathbf{a}}_{k}\breve{\psi} = -\hat{\mathbf{a}}_{k}\hat{R}\breve{\psi}\,, \tag{13}$$

$$\hat{R}\hat{\mathbf{e}}_{k}\breve{\psi} = \hat{\mathbf{e}}_{k}\hat{R}\breve{\psi}\,. \tag{14}$$

3. The System of Octonic First-Order Equations for Quantum Fields

Formally the octonic relativistic second-order equation[17] corresponding to the Einstein relation for energy and momentum

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = -\frac{m^2c^2}{\hbar^2}\breve{\psi} \tag{15}$$

is similar to the octonic wave equation[16] for the potentials of electromagnetic field. Therefore in the relativistic octonic quantum mechanics we can define some quantum fields, which will satisfy the first-order equations analogous to the Maxwell equations.

Indeed, by means of operator $\hat{R}$, which anticommutes with operator $\vec{\nabla}$ equation (15) can be represented in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,, \tag{16}$$

where the left part is the octonic product of two octonic operators. The representation (16) allows one to define quantum fields, which satisfy a system of first-order equations. The derivation of first-order equations is analogous to the procedure of obtaining the Maxwell equations in the octonic electrodynamics.[16] Let us consider the sequential action of operators in (16). After the action of the first operator we obtain the following expression:

$$\begin{aligned}
&\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\left(\varphi_0 + \vec{\tilde{\varphi}} + \tilde{\chi}_0 + \vec{\chi}\right)\\
&\quad = \frac{1}{c}\frac{\partial\varphi_0}{\partial t} + \frac{1}{c}\frac{\partial\vec{\tilde{\varphi}}}{\partial t} + \frac{1}{c}\frac{\partial\tilde{\chi}_0}{\partial t} + \frac{1}{c}\frac{\partial\vec{\chi}}{\partial t} + \vec{\nabla}\varphi_0 + (\vec{\nabla}\cdot\vec{\tilde{\varphi}})\\
&\qquad + [\vec{\nabla}\times\vec{\tilde{\varphi}}] + \vec{\nabla}\tilde{\chi}_0 + (\vec{\nabla}\cdot\vec{\chi}) + [\vec{\nabla}\times\vec{\chi}]\\
&\qquad + i\frac{mc}{\hbar}\varphi_0 + i\frac{mc}{\hbar}\vec{\tilde{\varphi}} - i\frac{mc}{\hbar}\tilde{\chi}_0 - i\frac{mc}{\hbar}\vec{\chi}\,,
\end{aligned} \tag{17}$$

which enables definition of quantum fields on the basis of the wave function. (The operations of scalar $(\cdot)$ and vector $[\times]$ octonic multiplication were considered in

Ref. 16). We will indicate these fields by index ψ:

$$e_\psi = \frac{1}{c}\frac{\partial\varphi_0}{\partial t} + (\vec{\nabla}\cdot\vec{\chi}) + i\frac{mc}{\hbar}\varphi_0\,, \tag{18}$$

$$\vec{E}_\psi = -\vec{\nabla}\varphi_0 - \frac{1}{c}\frac{\partial\vec{\chi}}{\partial t} + i\frac{mc}{\hbar}\vec{\chi} - \big[\vec{\nabla}\times\overleftrightarrow{\varphi}\big]\,, \tag{19}$$

$$\tilde{h}_\psi = \frac{i}{c}\frac{\partial\tilde{\chi}_0}{\partial t} + i\big(\vec{\nabla}\cdot\overleftrightarrow{\varphi}\big) + \frac{mc}{\hbar}\tilde{\chi}_0\,, \tag{20}$$

$$\overleftrightarrow{H}_\psi = -i\big[\vec{\nabla}\times\vec{\chi}\big] - i\vec{\nabla}\tilde{\chi}_0 - \frac{i}{c}\frac{\partial\overleftrightarrow{\varphi}}{\partial t} + \frac{mc}{\hbar}\overleftrightarrow{\varphi}\,. \tag{21}$$

Here e_ψ is an intensity of scalar field, $\vec{E}_\psi$ is an intensity of vector field, $\tilde{h}_\psi$ is an intensity of pseudoscalar field and $\overleftrightarrow{H}_\psi$ is an intensity of pseudovector field. Using fields definition, expression (17) can be rewritten in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = e_\psi - \vec{E}_\psi - i\tilde{h}_\psi + i\overleftrightarrow{H}_\psi\,. \tag{22}$$

Then from (16) we get the equation for the quantum fields

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right)\big(e_\psi - \vec{E}_\psi - i\tilde{h}_\psi + i\overleftrightarrow{H}_\psi\big) = 0\,. \tag{23}$$

Performing octonic multiplication and separating scalar, pseudoscalar, vector and pseudovector parts we obtain the system of first-order equations analogous to the Maxwell equations:

$$(\vec{\nabla}\cdot\vec{E}_\psi) = -\frac{1}{c}\frac{\partial e_\psi}{\partial t} + i\frac{mc}{\hbar}e_\psi \qquad \text{— is scalar part}\,, \tag{24}$$

$$\big(\vec{\nabla}\cdot\overleftrightarrow{H}_\psi\big) = -\frac{1}{c}\frac{\partial\tilde{h}_\psi}{\partial t} - i\frac{mc}{\hbar}\tilde{h}_\psi \qquad \text{— is pseudoscalar part}\,, \tag{25}$$

$$\big[\vec{\nabla}\times\overleftrightarrow{H}_\psi\big] = \frac{i}{c}\frac{\partial\vec{E}_\psi}{\partial t} + i\vec{\nabla}e_\psi - \frac{mc}{\hbar}\vec{E}_\psi \qquad \text{— is vector part}\,, \tag{26}$$

$$[\vec{\nabla}\times\vec{E}_\psi] = -\frac{i}{c}\frac{\partial\overleftrightarrow{H}_\psi}{\partial t} - i\vec{\nabla}\tilde{h}_\psi - \frac{mc}{\hbar}\overleftrightarrow{H}_\psi \qquad \text{— is pseudovector part}\,. \tag{27}$$

This system is absolutely equivalent to Eq. (16).

If we restrict the class of wave functions we can construct two-vector first-order equations for fields $\vec{E}_\psi$ and $\overleftrightarrow{H}_\psi$ only. At that the conditions $e_\psi = 0$ and $\tilde{h}_\psi = 0$ lead to the following "gauge" relations for the wave function:

$$\frac{1}{c}\frac{\partial\varphi_0}{\partial t} + (\vec{\nabla}\cdot\vec{\chi}) + i\frac{mc}{\hbar}\varphi_0 = 0\,, \tag{28}$$

$$\frac{1}{c}\frac{\partial\tilde{\chi}_0}{\partial t} + \big(\vec{\nabla}\cdot\overleftrightarrow{\varphi}\big) - i\frac{mc}{\hbar}\tilde{\chi}_0 = 0\,. \tag{29}$$

Under the conditions (28), (29) the equations for quantum fields can be written as

$$(\vec{\nabla} \cdot \vec{E}_\psi) = 0\,, \tag{30}$$

$$\left(\vec{\nabla} \cdot \overleftrightarrow{H}_\psi\right) = 0\,, \tag{31}$$

$$\left[\vec{\nabla} \times \overleftrightarrow{H}_\psi\right] = \frac{i}{c}\frac{\partial \vec{E}_\psi}{\partial t} - \frac{mc}{\hbar}\vec{E}_\psi\,, \tag{32}$$

$$[\vec{\nabla} \times \vec{E}_\psi] = -\frac{i}{c}\frac{\partial \overleftrightarrow{H}_\psi}{\partial t} - \frac{mc}{\hbar}\overleftrightarrow{H}_\psi\,. \tag{33}$$

In contrast to Proca equations[18–20] (enclosing field's intensities and potentials) which describe massive particles with spin 1 (see also generalized theory of integer spin in Refs. 21–23), the system (30)–(33) consists of equations for the field's intensities only and describes particles with spin 1/2. Note that if we take in (30)–(33) the mass equal to zero and choose the wave function as the four component potential of electromagnetic field then the system (30)–(33) will coincide with the Maxwell equations, and the condition (28) will coincide with the Lorentz gauge.

The system of equations (24)–(27) can be generalized for a particle in an external electromagnetic field. In this case we have to change operators in (16) by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + i\frac{e}{\hbar}\varPhi\,, \qquad \vec{\nabla} \to \vec{\nabla} - i\frac{e}{\hbar c}\vec{A}\,. \tag{34}$$

Then we obtain the equation

$$\begin{aligned}&\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\varPhi - \vec{\nabla} + i\frac{e}{\hbar c}\vec{A} - i\frac{mc}{\hbar}\hat{R}\right)\\ &\qquad \times \left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\varPhi + \vec{\nabla} - i\frac{e}{\hbar c}\vec{A} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,.\end{aligned} \tag{35}$$

After the action of the first operator we have

$$\begin{aligned}&\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\varPhi + \vec{\nabla} - i\frac{e}{\hbar c}\vec{A} + i\frac{mc}{\hbar}\hat{R}\right)(\varphi_0 + \overleftrightarrow{\varphi} + \tilde{\chi}_0 + \vec{\chi})\\ &= \frac{1}{c}\frac{\partial \varphi_0}{\partial t} + \frac{1}{c}\frac{\partial \overleftrightarrow{\varphi}}{\partial t} + \frac{1}{c}\frac{\partial \tilde{\chi}_0}{\partial t} + \frac{1}{c}\frac{\partial \vec{\chi}}{\partial t} + i\frac{e}{\hbar c}\varPhi\varphi_0 + i\frac{e}{\hbar c}\varPhi\overleftrightarrow{\varphi}\\ &\quad + i\frac{e}{\hbar c}\varPhi\tilde{\chi}_0 + i\frac{e}{\hbar c}\varPhi\vec{\chi} + \vec{\nabla}\varphi_0 + \left(\vec{\nabla} \cdot \overleftrightarrow{\varphi}\right) + \left[\vec{\nabla} \times \overleftrightarrow{\varphi}\right]\\ &\quad + \vec{\nabla}\tilde{\chi}_0 + (\vec{\nabla} \cdot \vec{\chi}) + [\vec{\nabla} \times \vec{\chi}] - i\frac{e}{\hbar c}\vec{A}\varphi_0 - i\frac{e}{\hbar c}(\vec{A} \cdot \overleftrightarrow{\varphi})\\ &\quad - i\frac{e}{\hbar c}[\vec{A} \times \overleftrightarrow{\varphi}] - i\frac{e}{\hbar c}\vec{A}\tilde{\chi}_0 - i\frac{e}{\hbar c}(\vec{A} \cdot \vec{\chi}) - i\frac{e}{\hbar c}[\vec{A} \times \vec{\chi}]\\ &\quad + i\frac{mc}{\hbar}\varphi_0 + i\frac{mc}{\hbar}\overleftrightarrow{\varphi} - i\frac{mc}{\hbar}\tilde{\chi}_0 - i\frac{mc}{\hbar}\vec{\chi}\,.\end{aligned} \tag{36}$$

In this case the quantum fields can be defined as

$$e_\psi = \frac{1}{c}\frac{\partial\varphi_0}{\partial t} + (\vec{\nabla}\cdot\vec{\chi}) + i\frac{mc}{\hbar}\varphi_0 + i\frac{e}{\hbar c}\Phi\varphi_0 - i\frac{e}{\hbar c}(\vec{A}\cdot\vec{\chi})\,, \tag{37}$$

$$\begin{aligned}\vec{E}_\psi = {} & -\vec{\nabla}\varphi_0 - \frac{1}{c}\frac{\partial\vec{\chi}}{\partial t} + i\frac{mc}{\hbar}\vec{\chi} - \big[\vec{\nabla}\times\overleftrightarrow{\varphi}\big] \\ & - i\frac{e}{\hbar c}\Phi\vec{\chi} + i\frac{e}{\hbar c}\varphi_0\vec{A} + i\frac{e}{\hbar c}\big[\vec{A}\times\overleftrightarrow{\varphi}\big]\,,\end{aligned} \tag{38}$$

$$\tilde{h}_\psi = i\frac{1}{c}\frac{\partial\tilde{\chi}_0}{\partial t} + i\big(\vec{\nabla}\cdot\overleftrightarrow{\varphi}\big) + \frac{mc}{\hbar}\tilde{\chi}_0 - \frac{e}{\hbar c}\Phi\tilde{\chi}_0 + \frac{e}{\hbar c}\big(\vec{A}\cdot\overleftrightarrow{\varphi}\big)\,, \tag{39}$$

$$\begin{aligned}\overleftrightarrow{H}_\psi = {} & -i[\vec{\nabla}\times\vec{\chi}] - i\vec{\nabla}\tilde{\chi}_0 - i\frac{1}{c}\frac{\partial\overleftrightarrow{\varphi}}{\partial t} + \frac{mc}{\hbar}\overleftrightarrow{\varphi} \\ & + \frac{e}{\hbar c}\Phi\overleftrightarrow{\varphi} - \frac{e}{\hbar c}[\vec{A}\times\vec{\chi}] - \frac{e}{\hbar c}\tilde{\chi}_0\vec{A}\,.\end{aligned} \tag{40}$$

Then (36) can be represented as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\Phi + \vec{\nabla} - i\frac{e}{\hbar c}\vec{A} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = e_\psi - \vec{E}_\psi - i\tilde{h}_\psi + i\overleftrightarrow{H}_\psi \tag{41}$$

and Eq. (35) for the quantum fields takes the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\Phi - \vec{\nabla} + i\frac{e}{\hbar c}\vec{A} - i\frac{mc}{\hbar}\hat{R}\right)\big(e_\psi - \vec{E}_\psi - i\tilde{h}_\psi + i\overleftrightarrow{H}_\psi\big) = 0\,. \tag{42}$$

Performing the octonic multiplication in (42) and separating scalar, pseudoscalar, vector and pseudovector parts we obtain the system of the equations for quantum field's intensities

$$(\vec{\nabla}\cdot\vec{E}_\psi) = -\frac{1}{c}\frac{\partial e_\psi}{\partial t} - i\frac{e}{\hbar c}\Phi e_\psi + i\frac{e}{\hbar c}(\vec{A}\cdot\vec{E}_\psi) + i\frac{mc}{\hbar}e_\psi\,, \tag{43}$$

$$\begin{aligned}\big[\vec{\nabla}\times\overleftrightarrow{H}_\psi\big] = {} & i\frac{1}{c}\frac{\partial\vec{E}_\psi}{\partial t} - \frac{e}{\hbar c}\Phi\vec{E}_\psi + i\vec{\nabla}e_\psi + \frac{e}{\hbar c}\vec{A}e_\psi \\ & + i\frac{e}{\hbar c}\big[\vec{A}\times\overleftrightarrow{H}_\psi\big] - \frac{mc}{\hbar}\vec{E}_\psi\,,\end{aligned} \tag{44}$$

$$\big(\vec{\nabla}\cdot\overleftrightarrow{H}_\psi\big) = -\frac{1}{c}\frac{\partial\tilde{h}_\psi}{\partial t} - i\frac{e}{\hbar c}\Phi\tilde{h}_\psi + i\frac{e}{\hbar c}\big(\vec{A}\cdot\overleftrightarrow{H}_\psi\big) - i\frac{mc}{\hbar}\tilde{h}_\psi\,, \tag{45}$$

$$\begin{aligned}[\vec{\nabla}\times\vec{E}_\psi] = {} & -i\frac{1}{c}\frac{\partial\overleftrightarrow{H}_\psi}{\partial t} + \frac{e}{\hbar c}\Phi\overleftrightarrow{H}_\psi - i\vec{\nabla}\tilde{h}_\psi \\ & + i\frac{e}{\hbar c}[\vec{A}\times\vec{E}_\psi] - \frac{e}{\hbar c}\vec{A}\tilde{h}_\psi - \frac{mc}{\hbar}\overleftrightarrow{H}_\psi\,.\end{aligned} \tag{46}$$

The system (43)–(46) is absolutely equivalent to the octonic equation (35).

Note that on the basis of systems (24)–(27), (30)–(33) and (43)–(46) one can obtain the quadratic forms analogous to the relations for energy and momentum as well as for Lorenz invariants of electromagnetic field.[16]

4. Octonic First-Order Equation

As it was shown in Ref. 17 the spin interaction of the particle with electromagnetic field can be described by the octonic second-order equation. At that in contrast to the Dirac theory the terms describing the interaction of spin 1/2 with electric and magnetic fields are appeared in the octonic second-order equation as a result of octonic multiplication without attraction of the first-order equation. However, in octonic quantum mechanics we can also construct the Dirac's-like first-order equations.

Let us turn to the octonic equation (15):

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = -\frac{m^2c^2}{\hbar^2}\breve{\psi}\,. \tag{47}$$

In this equation we can formally denote the result of action of one operator on function $\breve{\psi}$ as some new octonic function $\breve{W}$:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = -\frac{mc}{\hbar}\breve{W}\,. \tag{48}$$

Then the second-order equation (47) is equivalent to the system of two first-order equations:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = -\frac{mc}{\hbar}\breve{W}\,, \tag{49}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right)\breve{W} = \frac{mc}{\hbar}\breve{\psi}\,. \tag{50}$$

Acting on Eq. (50) by the operator of spatial inversion $\hat{R}$ we get

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = \frac{mc}{\hbar}(-\breve{W})\,, \tag{51}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\hat{R}\breve{W} = \frac{mc}{\hbar}\hat{R}\breve{\psi}\,. \tag{52}$$

On some conditions Eqs. (51), (52) can be absolutely equivalent. For that functions $\breve{W}$ and $\breve{\psi}$ should satisfy the following relations:

$$\breve{\psi} = \eta\hat{R}\breve{W}\,, \tag{53}$$

$$-\breve{W} = \eta\hat{R}\breve{\psi}\,, \tag{54}$$

where η is some constant. In particular for scalar η we obtain $\eta = \pm i$. So if the accessory function $\breve{W}$ satisfies the condition

$$\breve{W} = \pm i\hat{R}\breve{\psi}\,, \tag{55}$$

then the wave function $\breve{\psi}$ satisfies the first-order equation. The sign in (55) can be chosen arbitrarily. If $\breve{W} = +i\hat{R}\breve{\psi}$ the first-order equation is

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,. \tag{56}$$

Note that we can also act by $\hat{R}$ on Eq. (49). Then we get an equation with other sign before the gradient operator. Thus the octonic first-order equation can be written in four different forms:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,, \tag{57}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,, \tag{58}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,, \tag{59}$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,. \tag{60}$$

Note that the octon's algebra[17] is isomorphic to the Dirac matrixes algebra and Eqs. (57)–(60) are analogous to the Dirac equation.

Each Eqs. (57)–(60) is equivalent to the system of eight scalar equations for the components of the octonic wave function. If we search the solution to (57)–(60) as the plane wave

$$\breve{\psi} \sim \exp\left\{\frac{i}{\hbar}(-Et + p_x x + p_y y + p_z z)\right\}, \tag{61}$$

then the dispersion relation is

$$(E^2 - p^2c^2 - m^2c^4)^4 = 0\,, \tag{62}$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$. The roots of Eq. (62) $E = \pm\sqrt{p^2c^2 + m^2c^4}$ are fourthly degenerate.

There is also the inverse procedure of obtaining the second-order equation analogous to procedure used in the Dirac theory. For example, acting on Eq. (57) by operator

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right), \tag{63}$$

we get the following equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla} - i\frac{mc}{\hbar}\hat{R}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,. \tag{64}$$

However, we specially emphasize though Eq. (64) coincides in form with the second-order equation (16), but the solutions to (64) should also satisfy the first-order equation (57) simultaneously. The similar procedure can be specified for any Eq. (57)–(60).

Equations (57)–(60) can be generalized for a particle in the electromagnetic field using substitutions (34). For example in the presence of electromagnetic field

Eq. (57) can be rewritten in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + i\frac{e}{\hbar c}\Phi + \vec{\nabla} - i\frac{e}{\hbar c}\vec{A} + i\frac{mc}{\hbar}\hat{R}\right)\breve{\psi} = 0\,. \tag{65}$$

The first-order equations have an interesting interpretation. Since the operators in Eqs. (57) and (65) coincide with operators used in expressions (17) and (36) for the definition of quantum field's intensities (18)–(21) and (37)–(40) (see also (22) and (41)), then in fact the first-order equations (57) and (65) describe the particles, which do not have the quantum fields e_ψ, $\vec{E}_\psi$, $\tilde{h}_\psi$, $\overleftrightarrow{H}_\psi$.

5. Conclusion

We showed that in the frames of octonic quantum mechanics the second-order wave equation describing the particles with spin 1/2 can be reformulated in the form of the system of the first-order Maxwell's-like equations for the quantum fields. It was demonstrated that for the special class of wave functions the octonic second-order equation can be reduced to the single octonic first-order equation analogous to the Dirac equation. At the same time it was shown that the Dirac's-like first-order equations describe particles, which do not have quantum fields.

Thus we showed that particles with spin 1/2 can be described by two different (first-order and second-order) octonic equations. We assume that these equations describe the particles of different types. Probably the second-order equation tolerating the quantum fields introduction describes the baryons while the first-order equation describes the leptons.

Acknowledgments

The authors are very thankful to G. V. Mironova for kind assistance and moral support. Special thanks to the reviewer for the very useful comments.

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