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SEDEONIC GENERALIZATION OF RELATIVISTIC QUANTUM MECHANICS

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We represent sixteen-component values "sedeons," generating associative noncommutative space–time algebra. We demonstrate a generalization of relativistic quantum mechanics using sedeonic wave functions and sedeonic space–time operators. It is shown that the sedeonic second-order equation for the sedeonic wave function, obtained from the Einstein relation for energy and momentum, describes particles with spin 1/2. We showed that the sedeonic second-order wave equation can be reformulated in the form of the system of the first-order Maxwell-like equations for the massive fields. We proposed the sedeonic first-order equations analogous to the Dirac equation, which differ in space–time properties and describe several types of massive and massless particles. In particular we proposed four different equations, which could describe four types of neutrinos.

Keywords: Sedeons; space–time Clifford algebra; sedeonic Maxwell and Dirac equations; sedeonic neutrino equations.

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1. Introduction

It is known that scalar Schrödinger and Klein–Gordon equations for scalar wave function do not describe spin properties of quantum particles.[1,2] For the spin description W. Pauli and P. A. M. Dirac proposed matrix equations for the multicomponent spinor wave functions.[3,4] In the latter years many authors considered alternative possibilities to describe quantum particles by multicomponent wave functions on the basis of various systems of hypercomplex numbers.[5–23] The simplest generalizations of quantum mechanics based on quaternionic wave functions with spatial structure enclosing scalar and vector components were made in Refs. 5–11. However the essential imperfection of the quaternionic algebra is that the quaternions do not include pseudoscalar and pseudovector components. The consideration of total symmetry with respect to spatial inversion leads us to the eight-component wave functions enclosing scalar, pseudoscalar, vector and pseudovector components. However attempts to describe relativistic particles by means of different eight-component hypernumbers such as biquaternions,[4–15] octonions[16–20] and multivectors generating associative Clifford algebras[21–23] have not made appreciable progress. In particularly, the few attempts to describe relativistic particles by means of octonion wave functions are confronted by difficulties connected with octonions nonassociativity.[18] Moreover all systems of hypercomplex numbers, which have been applied up to now for the generalization of quantum mechanics (quaternions, biquaternions, octonions and multivectors) are the objects of hypercomplex space and do not have any consistent space-geometric interpretation. Recently we proposed eight-component values "octons"[24–26] generating a closed noncommutative associative algebra and having a clear well-defined geometric interpretation. It was shown that equations of relativistic quantum mechanics can be adequately generalized on the basis of octonic wave functions and octonic spatial operators. However all the above-mentioned eight-component wave functions do not describe the properties of quantum particles concerned with time transformation. The consideration of total space–time symmetry requires sixteen-component wave functions.

There are some publications describing the attempts to develop quantum mechanics using different sixteen-component hypernumbers. In particular, one of approaches is the application of hypernumbers sedenions, which are obtained from octonions by Cayley–Dickson extension procedure.[27–30] But as in the case of octonions the essential imperfection of sedenions is their nonassociativity. Another approach is the description of quantum particles by hypercomplex multivectors generating associative space–time Clifford algebras. The basic idea of such multivectors is an introduction of additional noncommutative time unit, which is orthogonal to the space units.[31,32] However the application of such multivectors in quantum mechanics is considered in general as one of abstract algebraic scheme which enables the reformulation of Dirac equation in terms of nonspinor wave functions but does not touch the physical entity of this equation.

In this paper we represent sixteen-component values "sedeons," which are the generalization of the proposed previously "octons" and generate associative noncommutative space–time algebra. On the basis of sedeonic wave functions and sedeonic space–time operators the generalized equations of relativistic quantum mechanics are formulated. We show that sedeonic second-order and first-order equations differing in space–time properties enable the consideration of several types of massive and massless fields.

2. Algebra of Sedeons

Let us consider four groups of values, which are differed with respect to spatial and time inversion.

(1) Absolute scalars (A_0) and absolute vectors ($\vec{A}$) are not transformed under spatial and time inversion.

(2) Space scalars (B_{0r}) and space vectors ($\vec{B}_r$) are changed (in sign) under spatial inversion and are not transformed under time inversion.
(3) Time scalars (C_{0t}) and time vectors ($\vec{C}_t$) are changed under time inversion and are not transformed under spatial inversion.
(4) Space–time scalars (D_{0rt}) and space–time vectors ($\vec{D}_{rt}$) are changed under spatial and time inversion.

Here indexes r and t indicate the transformations (r for spatial inversion and t for time inversion), which change the corresponding values. Let us formally define the operation of spatial inversion and time inversion, which are realized by the operators $\hat{I}_r$ and $\hat{I}_t$. These operators change the sign of corresponding values:

$$\begin{aligned}\hat{I}_r :\ & A_0,\ \vec{A},\ B_{0r},\ \vec{B}_r,\ C_{0t},\ \vec{C}_t,\ D_{0rt},\ \vec{D}_{rt}\\ & \Rightarrow A_0,\ \vec{A},\ -B_{0r},\ -\vec{B}_r,\ C_{0t},\ \vec{C}_t,\ -D_{0rt},\ -\vec{D}_{rt}\,;\\ \hat{I}_t :\ & A_0,\ \vec{A},\ B_{0r},\ \vec{B}_r,\ C_{0t},\ \vec{C}_t,\ D_{0rt},\ \vec{D}_{rt}\\ & \Rightarrow A_0,\ \vec{A},\ B_{0r},\ \vec{B}_r,\ -C_{0t},\ -\vec{C}_t,\ -D_{0rt},\ -\vec{D}_{rt}\,.\end{aligned}$$

All introduced values can be integrated into one space–time object. For this purpose we propose the special sixteen-component values, which will be named "sedeons" (in contrast to sedenions).

The sixteen-component sedeon $\tilde{\mathbf{W}}$ is defined by the following expression:

$$\tilde{\mathbf{W}} = A_0 + \vec{A} + B_{0r} + \vec{B}_r + C_{0t} + \vec{C}_t + D_{0rt} + \vec{D}_{rt}\,. \tag{1}$$

The sedeon (1) can be written also in the expanded form

$$\begin{aligned}\tilde{\mathbf{W}} = {} & A_0\mathbf{e} + A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3 + B_\mathbf{0}\mathbf{e_r} + B_1\mathbf{a_{1r}} + B_2\mathbf{a_{2r}} + B_3\mathbf{a_{3r}} + C_\mathbf{0}\mathbf{e_t}\\ & + C_1\mathbf{a_{1t}} + C_2\mathbf{a_{2t}} + C_3\mathbf{a_{3t}} + D_\mathbf{0}\mathbf{e_{rt}} + D_1\mathbf{a_{1rt}} + D_2\mathbf{a_{2rt}} + D_3\mathbf{a_{3rt}}\,,\end{aligned} \tag{2}$$

where values $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are absolute unit vectors; $\mathbf{a_{1r}}$, $\mathbf{a_{2r}}$ and $\mathbf{a_{3r}}$ are space unit vectors; $\mathbf{a_{1t}}$, $\mathbf{a_{2t}}$ and $\mathbf{a_{3t}}$ are time unit vectors; $\mathbf{a_{1rt}}$, $\mathbf{a_{2rt}}$ and $\mathbf{a_{3rt}}$ are space–time unit vectors; $\mathbf{e}$ is absolute scalar unit ($\mathbf{e} \equiv 1$); $\mathbf{e_r}$ is space scalar unit; $\mathbf{e_t}$ is time scalar unit; $\mathbf{e_{rt}}$ is space–time scalar unit. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$; $\mathbf{a_{1r}}$, $\mathbf{a_{2r}}$, $\mathbf{a_{3r}}$; $\mathbf{a_{1t}}$, $\mathbf{a_{2t}}$, $\mathbf{a_{3t}}$ and $\mathbf{a_{1rt}}$, $\mathbf{a_{2rt}}$, $\mathbf{a_{3rt}}$ be the right Cartesian bases and corresponding unit vectors are parallel to each other. The sedeonic components A_s, B_s, C_s, D_s ($s = 0, 1, 2, 3$) are numbers (complex in general). The values

$$1,\ \mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{e_r},\ \mathbf{a_{1r}},\ \mathbf{a_{2r}},\ \mathbf{a_{3r}},\ \mathbf{e_t},\ \mathbf{a_{1t}},\ \mathbf{a_{2t}},\ \mathbf{a_{3t}},\ \mathbf{e_{rt}},\ \mathbf{a_{1rt}},\ \mathbf{a_{2rt}},\ \mathbf{a_{3rt}} \tag{3}$$

are the space–time basis of sedeon. The rules of multiplication of basis elements (3) are formulated taking into account the symmetry of their products with respect to the operations of spatial and time inversion.

The squares of sedeonic scalar units and unit vectors are positively defined and equal to 1:

$$\mathbf{a}_j^2 = \mathbf{e_r}^2 = \mathbf{a}_{j\mathbf{r}}^2 = \mathbf{e_t}^2 = \mathbf{a}_{j\mathbf{t}}^2 = \mathbf{e_{rt}}^2 = \mathbf{a}_{j\mathbf{rt}}^2 = \mathbf{1} \quad (j = 1, 2, 3)\,. \tag{4}$$

Table 1.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2.

	$\mathbf{e}_\mathbf{r}$	$\mathbf{e}_\mathbf{t}$	$\mathbf{e}_\mathbf{rt}$
$\mathbf{e}_\mathbf{r}$	1	$\mathbf{e}_\mathbf{rt}$	$\mathbf{e}$
$\mathbf{e}_\mathbf{t}$	$\mathbf{e}_\mathbf{rt}$	1	$\mathbf{e}_\mathbf{r}$
$\mathbf{e}_\mathbf{rt}$	$\mathbf{e}_\mathbf{t}$	$\mathbf{e}_\mathbf{r}$	1

The units $\mathbf{e}_\mathbf{r}$, $\mathbf{e}_\mathbf{t}$, $\mathbf{e}_\mathbf{rt}$ commute with each other and with all sedeonic unit vectors. All unit vectors can be expressed through the absolute vectors:

$$\mathbf{a}_{j\mathbf{r}} = \mathbf{e}_\mathbf{r}\mathbf{a}_j\,, \quad \mathbf{a}_{j\mathbf{t}} = \mathbf{e}_\mathbf{t}\mathbf{a}_j\,, \quad \mathbf{a}_{j\mathbf{rt}} = \mathbf{e}_\mathbf{rt}\mathbf{a}_j\,. \tag{5}$$

The absolute unit vectors anticommute with each other:

$$\mathbf{a}_k\mathbf{a}_j = -\mathbf{a}_j\mathbf{a}_k \quad (k \neq j\,,\ k = 1,2,3)\,. \tag{6}$$

The commutation rules for the rest unit vectors are the same and follow directly from relations (5).

The rules of multiplication are constructed taking into account (4)–(6). For example, the multiplication rules for the absolute unit vectors are

$$\mathbf{a}_1\mathbf{a}_2 = i\mathbf{a}_3\,, \quad \mathbf{a}_2\mathbf{a}_3 = i\mathbf{a}_1\,, \quad \mathbf{a}_3\mathbf{a}_1 = i\mathbf{a}_2\,,$$

where the value i is the imaginary unit ($i^2 = -1$). All multiplication and commutation rules can be represented by means of two simple tables describing multiplication of absolute unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and sedeonic units $\mathbf{e}_\mathbf{r}$, $\mathbf{e}_\mathbf{t}$, $\mathbf{e}_\mathbf{rt}$ (see Tables 1 and 2).

We would like to emphasize especially that sedeonic algebra is associative. The property of associativity follows directly from multiplication rules.

Thus the sedeon $\tilde{\mathbf{W}}$ is the complicated space–time object consisting of absolute scalar, space scalar, time scalar, space–time scalar, absolute vector, space vector, time vector and space–time vector. Note that 1, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ is distinguish sedeonic basis since the corresponding components of sedeon are not transformed under spatial and time inversion. Taking into account the relations between different elements of sedeonic basis the sedeon can be represented in the compact form. Introducing the designations of sedeon-scalars

$$\mathbf{W}_0 = A_0 + B_0\mathbf{e}_\mathbf{r} + C_0\mathbf{e}_\mathbf{t} + D_0\mathbf{e}_\mathbf{rt}\,, \tag{7}$$

$$\mathbf{W}_1 = A_1 + B_1\mathbf{e}_\mathbf{r} + C_1\mathbf{e}_\mathbf{t} + D_1\mathbf{e}_\mathbf{rt}\,, \tag{8}$$

$$\mathbf{W}_2 = A_2 + B_2\mathbf{e}_\mathbf{r} + C_2\mathbf{e}_\mathbf{t} + D_2\mathbf{e}_\mathbf{rt}\,, \tag{9}$$

$$\mathbf{W}_3 = A_3 + B_3\mathbf{e}_\mathbf{r} + C_3\mathbf{e}_\mathbf{t} + D_3\mathbf{e}_\mathbf{rt}\,, \tag{10}$$

we can write the sedeon (2) as

$$\tilde{\mathbf{W}} = \mathbf{W}_0 + \mathbf{W}_1\mathbf{a}_1 + \mathbf{W}_2\mathbf{a}_2 + \mathbf{W}_3\mathbf{a}_3$$

or, introducing the sedeon-vector

$$\vec{\mathbf{W}} = \mathbf{W}_1\mathbf{a}_1 + \mathbf{W}_2\mathbf{a}_2 + \mathbf{W}_3\mathbf{a}_3\,,$$

we can represent the sedeon in a very compact form

$$\tilde{\mathbf{W}} = \mathbf{W}_0 + \vec{\mathbf{W}}\,.$$

Further we will indicate sedeon-scalars and sedeon-vectors with the bold capital letters.

Let us consider the rules of sedeonic multiplication in detail. In correspondence with rules of multiplication for sedeon basis elements the sedeonic product of two sedeons can be represented in the following form:

$$\begin{aligned}\tilde{\mathbf{W}}_1\tilde{\mathbf{W}}_2 &= (\mathbf{W}_{10} + \vec{\mathbf{W}}_1)(\mathbf{W}_{20} + \vec{\mathbf{W}}_2)\\ &= \mathbf{W}_{10}\mathbf{W}_{20} + \mathbf{W}_{10}\vec{\mathbf{W}}_2 + \mathbf{W}_{20}\vec{\mathbf{W}}_1\\ &\quad + (\vec{\mathbf{W}}_1 \cdot \vec{\mathbf{W}}_2) + [\vec{\mathbf{W}}_1 \times \vec{\mathbf{W}}_2]\,.\end{aligned} \tag{11}$$

Here we denoted the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol "$\cdot$" and round brackets

$$(\vec{\mathbf{W}}_1 \cdot \vec{\mathbf{W}}_2) = \mathbf{W}_{11}\mathbf{W}_{21} + \mathbf{W}_{12}\mathbf{W}_{22} + \mathbf{W}_{13}\mathbf{W}_{23}\,, \tag{12}$$

and sedeonic vector multiplication (external product) by symbol "$\times$" and square brackets,

$$\begin{aligned}[\vec{\mathbf{W}}_1 \times \vec{\mathbf{W}}_2] &= i(\mathbf{W}_{12}\mathbf{W}_{23} - \mathbf{W}_{13}\mathbf{W}_{22})\mathbf{a}_1 + i(\mathbf{W}_{13}\mathbf{W}_{21} - \mathbf{W}_{11}\mathbf{W}_{23})\mathbf{a}_2\\ &\quad + i(\mathbf{W}_{11}\mathbf{W}_{22} - \mathbf{W}_{12}\mathbf{W}_{21})\mathbf{a}_3\,.\end{aligned} \tag{13}$$

In (11)–(13) the component multiplication is performed in accordance with (7)–(10) and Table 2. Thus the sedeonic product

$$\tilde{\mathbf{F}} = \tilde{\mathbf{W}}_1\tilde{\mathbf{W}}_2 = \mathbf{F}_0 + \vec{\mathbf{F}}$$

has the following components:

$$\begin{aligned}\mathbf{F}_0 &= \mathbf{W}_{10}\mathbf{W}_{20} + \mathbf{W}_{11}\mathbf{W}_{21} + \mathbf{W}_{12}\mathbf{W}_{22} + \mathbf{W}_{13}\mathbf{W}_{23}\,,\\ \mathbf{F}_1 &= \mathbf{W}_{10}\mathbf{W}_{21} + \mathbf{W}_{20}\mathbf{W}_{11} + i\mathbf{W}_{12}\mathbf{W}_{23} - i\mathbf{W}_{13}\mathbf{W}_{22}\,,\\ \mathbf{F}_2 &= \mathbf{W}_{10}\mathbf{W}_{22} + \mathbf{W}_{20}\mathbf{W}_{12} + i\mathbf{W}_{13}\mathbf{W}_{21} - i\mathbf{W}_{11}\mathbf{W}_{23}\,,\\ \mathbf{F}_3 &= \mathbf{W}_{10}\mathbf{W}_{23} + \mathbf{W}_{20}\mathbf{W}_{13} + i\mathbf{W}_{11}\mathbf{W}_{22} - i\mathbf{W}_{12}\mathbf{W}_{21}\,.\end{aligned}$$

Let us also introduce the operators of space and time sedeonic conjugation ($\hat{R}_r$ and $\hat{R}_t$ respectively). For numerical sedeons these operators coincide with the operators of spatial and time inversion ($\hat{I}_r$ and $\hat{I}_t$). However for coordinate- and time-dependent sedeonic fields the operators $\hat{R}_r$ and $\hat{R}_t$ change the sign of corresponding components, but do not act on coordinates and time. The operator $\hat{R}_r$ changes the sign of space and space–time components of the sedeonic field $\mathbf{W}_s(\vec{r},t)$ ($s = 0, 1, 2, 3$; see (7)–(10)):

$$\hat{R}_r \mathbf{W}_s(\vec{r},t) = A_s(\vec{r},t) - B_s(\vec{r},t)\mathbf{e_r} + C_s(\vec{r},t)\mathbf{e_t} - D_s(\vec{r},t)\mathbf{e_{rt}}\,, \quad \hat{R}_r^2 = 1\,. \tag{14}$$

The operator $\hat{R}_t$ changes the sign of time and space–time components of the sedeonic field:

$$\hat{R}_t \mathbf{W}_s(\vec{r},t) = A_s(\vec{r},t) + B_s(\vec{r},t)\mathbf{e_r} - C_s(\vec{r},t)\mathbf{e_t} - D_s(\vec{r},t)\mathbf{e_{rt}}\,, \quad \hat{R}_t^2 = 1\,. \tag{15}$$

Also we can introduce the operator of space–time conjugation

$$\hat{R}_{rt} = \hat{R}_r \hat{R}_t\,,$$

which has the following property:

$$\hat{R}_{rt} \mathbf{W}_s(\vec{r},t) = A_s(\vec{r},t) - B_s(\vec{r},t)\mathbf{e_r} - C_s(\vec{r},t)\mathbf{e_t} + D_s(\vec{r},t)\mathbf{e_{rt}}\,, \quad \hat{R}_{rt}^2 = 1\,. \tag{16}$$

In the next sections we apply the sedeonic algebra to the generalization of relativistic quantum mechanics.

3. Sedeonic Second-Order Equations

Previously we proposed the octonic relativistic second-order equation[25] describing particles with spin 1/2. In addition the sedeon's algebra takes into account transformational properties of values with respect to time inversion. In this section by analogy with Ref. 25 we propose generalized sedeonic second-order equation and consider its space–time properties.

Let us consider the wave function of a relativistic particle in the form of a sixteen-component sedeon

$$\tilde{\mathbf{W}}(\vec{r},t) = \mathbf{W}_0(\vec{r},t) + \mathbf{W}_1(\vec{r},t)\mathbf{a}_1 + \mathbf{W}_2(\vec{r},t)\mathbf{a}_2 + \mathbf{W}_3(\vec{r},t)\mathbf{a}_3 \tag{17}$$

with components

$$\mathbf{W}_s(\vec{r},t) = A_s(\vec{r},t) + B_s(\vec{r},t)\mathbf{e_r} + C_s(\vec{r},t)\mathbf{e_t} + D_s(\vec{r},t)\mathbf{e_{rt}}\,. \tag{18}$$

The components $A_s(\vec{r},t)$, $B_s(\vec{r},t)$, $C_s(\vec{r},t)$ and $D_s(\vec{r},t)$ are scalar (complex in general) functions of spatial coordinates and time ($s = 0, 1, 2, 3$).

The wave function of a free particle should satisfy an equation, which is obtained from the Einstein relation between particle energy and momentum,

$$E^2 - c^2\vec{p}^2 = m^2c^4\,, \tag{19}$$

by means of changing classical momentum $\vec{p}$ and energy E on corresponding quantum-mechanical operators. In sedeonic algebra the quadratic form (19) can be represented in the following generalized form:

$$(E - \alpha c\vec{p})(E + \alpha c\vec{p}) = m^2c^4\,, \tag{20}$$

where α takes any meaning from the set $\{1, \pm\mathbf{e_r}, \pm\mathbf{e_t}, \pm\mathbf{e_{rt}}\}$. This equation enables introduction of several sedeonic wave equations differing in space–time properties. Further we consider various opportunities for the description of quantum systems on the base of different sedeonic space–time operators $\hat{\alpha}$.

Let us consider the operators of energy and momentum

$$\hat{E} = i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \hat{\vec{p}} = -i\hbar\vec{\nabla}\,,$$

where the gradient vector has the form

$$\vec{\nabla} = \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3\,.$$

Then we can formally write the generalized sedeonic wave equation obtained from (20) in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} = -\frac{m^2c^2}{\hbar^2}\tilde{\mathbf{W}}\,, \tag{21}$$

where c is the velocity of light, m is the mass of the particle and $\hbar$ is the Planck constant. Here the operator $\hat{\alpha}$ can take any meaning from ($\hat{\alpha} \in \{1, \pm\hat{\mathbf{e}}_\mathbf{r}, \pm\hat{\mathbf{e}}_\mathbf{t}, \pm\hat{\mathbf{e}}_\mathbf{rt}\}$). We also assume that the sedeonic wave function $\tilde{\mathbf{W}}$ is twice continuously differentiable, so $[\vec{\nabla}\times\vec{\nabla}]\tilde{\mathbf{W}} = 0$. Equation (21) can be written in the following expanded operator form:

$$\begin{aligned}&\left\{\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\left(\frac{\partial}{\partial x}\hat{\mathbf{a}}_1 + \frac{\partial}{\partial y}\hat{\mathbf{a}}_2 + \frac{\partial}{\partial z}\hat{\mathbf{a}}_3\right)\right\} \\ &\quad\times\left\{\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\left(\frac{\partial}{\partial x}\hat{\mathbf{a}}_1 + \frac{\partial}{\partial y}\hat{\mathbf{a}}_2 + \frac{\partial}{\partial z}\hat{\mathbf{a}}_3\right)\right\}\tilde{\mathbf{W}}(\vec{r},t) = -\frac{m^2c^2}{\hbar^2}\tilde{\mathbf{W}}(\vec{r},t)\,,\end{aligned} \tag{22}$$

where the space–time operators $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$ and $\hat{\alpha}$ in the left part of Eq. (22) transform the space–time structure of the wave function by means of sedeonic multiplication. For example, the action of the $\hat{\mathbf{a}}_3$ operator can be represented as sedeonic multiplication of unit vector $\mathbf{a}_3$ and sedeon $\tilde{\mathbf{W}}$:

$$\hat{\mathbf{a}}_3\tilde{\mathbf{W}} = \mathbf{a}_3\tilde{\mathbf{W}} = \mathbf{W}_3 - i\mathbf{W}_2\mathbf{a}_1 + i\mathbf{W}_1\mathbf{a}_2 + \mathbf{W}_0\mathbf{a}_3\,.$$

Further we will use symbolic designations $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$, $\hat{\mathbf{e}}_\mathbf{r}$, $\hat{\mathbf{e}}_\mathbf{t}$, $\hat{\mathbf{e}}_\mathbf{rt}$ in the operator part of equations but $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{e_r}$, $\mathbf{e_t}$ and $\mathbf{e_{rt}}$ designations in the wave functions. The rules of multiplication and commutation for space–time operators are analogues to the rules for corresponding elements of sedeonic basis.

To describe a particle in an external electromagnetic field the following change of quantum-mechanical operators should be made:[2]

$$\hat{E} \to \hat{E} - e\Phi\,, \quad \hat{\vec{p}} \to \hat{\vec{p}} - \frac{e}{c}\vec{A}\,, \tag{23}$$

where Φ is absolute scalar potential, $\vec{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3$ is absolute vector potential of the electromagnetic field, e is the particle charge ($e < 0$ for the electron). The change (23) is equivalent to the following change of differential operators:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{ie}{\hbar}\Phi\,, \quad \vec{\nabla} \to \vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\,. \tag{24}$$

Using substitution (24) we can write Eq. (21) as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi - \hat{\alpha}\vec{\nabla} + \frac{ie}{\hbar c}\hat{\alpha}\vec{A}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi + \hat{\alpha}\vec{\nabla} - \frac{ie}{\hbar c}\hat{\alpha}\vec{A}\right)\tilde{\mathbf{W}} = -\frac{m^2c^2}{\hbar^2}\tilde{\mathbf{W}}\,. \tag{25}$$

The multiplication of sedeonic operators in the left part of (25) leads us to the following equation:

$$\begin{aligned}\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{2ie}{\hbar c}\left((\vec{A}\cdot\vec{\nabla}) + \frac{\Phi}{c}\frac{\partial}{\partial t}\right) + \frac{m^2c^2}{\hbar^2} + \frac{e^2}{\hbar^2c^2}\left(A^2 - \Phi^2\right)\right]\tilde{\mathbf{W}} \\ - \frac{e}{\hbar c}\vec{H}\tilde{\mathbf{W}} + \frac{ie}{\hbar c}\hat{\alpha}\vec{E}\tilde{\mathbf{W}} = 0\,.\end{aligned} \tag{26}$$

Here we have taken into account that $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ is absolute vector of the electric field, $\vec{H} = -i[\vec{\nabla}\times\vec{A}]$ is absolute vector of the magnetic field. $(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0$ is the condition of the Lorentz gauge. Note that the sedeonic equation (26) encloses the specific terms $\frac{e}{\hbar c}\vec{H}\tilde{\mathbf{W}}$ and $\frac{ie}{\hbar c}\hat{\alpha}\vec{E}\tilde{\mathbf{W}}$, where the fields $\vec{E}$ and $\vec{H}$ play the role of spatial sedeonic operators. It is seen that in the presence of electric field the second-order equation (26) essentially depends on the space–time properties of operator $\hat{\alpha}$.

For a relativistic particle in an external homogeneous magnetic field directed along the Z axis

$$\vec{H} = B\mathbf{a}_3$$

the energy spectrum obtained from the solution to Eq. (26) is defined by the eigenvalue λ of the spatial operator $\mathbf{a}_3$ ($\lambda = \pm 1$) and has the form (the detailed derivation see in Ref. 25)

$$E^2_{n,\lambda} = m^2c^4 + p_z^2c^2 + |e|B\hbar c(2n+1) - \lambda eB\hbar c\,. \tag{27}$$

This set of energies is absolutely identical to the energy spectrum obtained from the relativistic second-order equation following from the Dirac equation.[2] The expression (27) allows one to state that eigenvalue λ of operator $\hat{\mathbf{a}}_3$ has the sense of spin projection and the second-order equation (26) correctly describes the interaction between spin 1/2 and the electromagnetic field.

If the wave function is the eigenfunction of the operator $\hat{\mathbf{a}}_3$, then some general statements about the spatial structure of the wave function can be made. In the stationary state with energy E the wave function can be represented in the following form:

$$\tilde{\mathbf{W}}_\lambda(\vec{r},t) = \left\{\mathbf{F}_1^{(\lambda)}(\vec{r})(1+\lambda\mathbf{a}_3) + \mathbf{F}_2^{(\lambda)}(\vec{r})(\mathbf{a}_1 + i\lambda\mathbf{a}_2)\right\}e^{-i\omega t}, \tag{28}$$

where $\omega = E/\hbar$, $\mathbf{F}_1^{(\lambda)}(\vec{r})$ and $\mathbf{F}_2^{(\lambda)}(\vec{r})$ are arbitrary sedeon-scalar functions. The wave function (28) is the space–time object, which we name sedeonic oscillator. The real and imaginary parts of the component $(1+\lambda\mathbf{a}_3)e^{-i\omega t}$ are a combination of an absolute vector directed parallel to the Z axis and an absolute scalar oscillating with the frequency ω. Here the phase difference between oscillations of scalar and vector parts equals 0 in case of $\lambda = 1$ or π in case of $\lambda = -1$. The real and imaginary parts of the component $(\mathbf{a}_1 + \lambda i\mathbf{a}_2)e^{-i\omega t}$ have the form of absolute vectors rotating in the plane perpendicular to the Z axis also with the frequency ω. The direction of the rotation depends on the sign of λ. When $\lambda = +1$ a vector of angular velocity is directed along the Z axis but when $\lambda = -1$ this vector has the opposite direction. The transformational properties of the wave function (30) are defined by sedeon-scalar functions $\mathbf{F}_1^{(\lambda)}(\vec{r})$ and $\mathbf{F}_2^{(\lambda)}(\vec{r})$.

4. Sedeonic Maxwell-like Equations for Massive Fields with Spin 1/2

On the basis of sedeonic wave functions we can define fields, which satisfy the first-order equations analogous to the Maxwell equations in electrodynamics.

From definitions (14)–(16) it is easy to see that the operator $\hat{R}_r$ anticommutes with $\hat{\mathbf{e}}_\mathbf{r}$, $\hat{\mathbf{e}}_\mathbf{rt}$ operators and commutes with $\hat{\mathbf{e}}_\mathbf{t}$, $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$, the operator $\hat{R}_t$ anticommutes with $\hat{\mathbf{e}}_\mathbf{t}$, $\hat{\mathbf{e}}_\mathbf{rt}$ and commutes with $\hat{\mathbf{e}}_\mathbf{r}$, $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$, and the operator $\hat{R}_{rt}$ anticommutes with $\hat{\mathbf{e}}_\mathbf{r}$, $\hat{\mathbf{e}}_\mathbf{t}$ and commutes with $\hat{\mathbf{e}}_\mathbf{rt}$, $\hat{\mathbf{a}}_1$, $\hat{\mathbf{a}}_2$, $\hat{\mathbf{a}}_3$. Moreover operators $\hat{R}_r$, $\hat{R}_t$ and $\hat{R}_{rt}$ commute with each other.

Let us introduce the generalized operator of conjugation $\hat{\rho}$, which takes the meaning from ($\hat{\rho} \in \{\pm\hat{R}_r, \pm\hat{R}_t, \pm\hat{R}_{rt}\}$). We also impose an additional important condition. Further we will consider the operator $\hat{\alpha}$, which takes the meaning only from the set $\{\pm\hat{\mathbf{e}}_\mathbf{r}, \pm\hat{\mathbf{e}}_\mathbf{t}, \pm\hat{\mathbf{e}}_\mathbf{rt}\}$ and suppose that the operators $\hat{\alpha}$ and $\hat{\rho}$ anticommute with each other in any expression, i.e. these operators satisfy the following condition:

$$\hat{\alpha}\hat{\rho}\tilde{\mathbf{W}} = -\hat{\rho}\hat{\alpha}\tilde{\mathbf{W}}.$$

For example, if we choose $\hat{\alpha} = \hat{\mathbf{e}}_\mathbf{r}$, then $\hat{\rho} = \hat{R}_r$ or $\hat{\rho} = \hat{R}_{rt}$, etc. Then using operators $\hat{\alpha}$ and $\hat{\rho}$ we can write generalized second-order equation (21) in the following operator form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla} - i\frac{mc}{\hbar}\hat{\rho}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = 0. \tag{29}$$

Let us consider the sequential action of operators in (29). After the action of the first operator we obtain

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = \frac{1}{c}\frac{\partial \mathbf{W}_0}{\partial t} + \frac{1}{c}\frac{\partial \vec{\mathbf{W}}}{\partial t} + \hat{\alpha}\vec{\nabla}\mathbf{W}_0 + \hat{\alpha}(\vec{\nabla}\cdot\vec{\mathbf{W}})$$
$$+ \hat{\alpha}[\vec{\nabla}\times\vec{\mathbf{W}}] + i\frac{mc}{\hbar}\hat{\rho}\mathbf{W}_0 + i\frac{mc}{\hbar}\hat{\rho}\vec{\mathbf{W}}\,. \tag{30}$$

Let us introduce the complex fields:

$$\mathbf{G}_0 = \frac{1}{c}\frac{\partial \mathbf{W}_0}{\partial t} + \hat{\alpha}(\vec{\nabla}\cdot\vec{\mathbf{W}}) + i\frac{mc}{\hbar}\hat{\rho}\mathbf{W}_0\,,$$
$$\vec{\mathbf{G}} = -\hat{\alpha}\vec{\nabla}\mathbf{W}_0 - \frac{1}{c}\frac{\partial \vec{\mathbf{W}}}{\partial t} - i\frac{mc}{\hbar}\hat{\rho}\vec{\mathbf{W}} - \hat{\alpha}[\vec{\nabla}\times\vec{\mathbf{W}}]\,.$$

Here $\mathbf{G}_0$ is a sedeon-scalar field and $\vec{\mathbf{G}}$ is sedeon-vector field. Using field's definition the expression (30) can be rewritten in the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = \mathbf{G}_0 - \vec{\mathbf{G}}\,. \tag{31}$$

Then Eq. (29) can be rewritten as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla} - i\frac{mc}{\hbar}\hat{\rho}\right)(\mathbf{G}_0 - \vec{\mathbf{G}}) = 0\,.$$

Performing sedeonic multiplication and separating sedeon-scalar and sedeon-vector parts we obtain the system of the first-order equations for field's intensities:

$$\begin{aligned}(\hat{\alpha}\vec{\nabla}\cdot\vec{\mathbf{G}}) &= -\frac{1}{c}\frac{\partial \mathbf{G}_0}{\partial t} + i\frac{mc}{\hbar}\hat{\rho}\mathbf{G}_0\,,\\ [\hat{\alpha}\vec{\nabla}\times\vec{\mathbf{G}}] &= \frac{1}{c}\frac{\partial \vec{\mathbf{G}}}{\partial t} + \hat{\alpha}\vec{\nabla}\mathbf{G}_0 - i\frac{mc}{\hbar}\hat{\rho}\vec{\mathbf{G}}\,.\end{aligned} \tag{32}$$

This system is absolutely equivalent to Eq. (29).

Equations (29) and (32) include the Maxwell equations for electromagnetic field in a vacuum as the special case. Indeed let us take the mass equal to zero and choose the wave function in the form of incomplete sedeon

$$\tilde{\mathbf{W}} = \mathit{\Phi} + \alpha\vec{A}\,,$$

where $\mathit{\Phi}$ is the absolute scalar potential, $\vec{A}$ is absolute vector potential and α is undefined constant ($\alpha \in \{\mathbf{e_r}, \mathbf{e_t}, \mathbf{e_{rt}}\}$) describing transformational properties of the vector potential. Then the generalized sedeonic equation of electrodynamics can be written in the following compact form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)(\mathit{\Phi} + \alpha\vec{A}) = 0\,. \tag{33}$$

Applying the operator in Eq. (33) to the sedeon of electromagnetic potentials, we get

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)(\Phi + \alpha\vec{A}) = \frac{1}{c}\frac{\partial\Phi}{\partial t} + \hat{\alpha}\vec{\nabla}\Phi + \alpha\frac{1}{c}\frac{\partial\vec{A}}{\partial t} + (\hat{\alpha}\vec{\nabla}\cdot\alpha\vec{A}) + [\hat{\alpha}\vec{\nabla}\times\alpha\vec{A}]\,. \tag{34}$$

For correct definition of electric and magnetic field we should require that α and $\hat{\alpha}$ have the same space–time properties, so $\hat{\alpha} = \alpha$ in this expression. Then electric and magnetic fields are defined in standard sedeonic form

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\Phi\,, \qquad \vec{H} = -i[\vec{\nabla}\times\vec{A}]\,.$$

Using the Lorentz gauge

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + (\vec{\nabla}\cdot\vec{A}) = 0\,,$$

we can rewrite the expression (34) in the following form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)(\Phi + \alpha\vec{A}) = i\vec{H} - \alpha\vec{E}\,.$$

Then sedeonic equation (33) can be written as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)(i\vec{H} - \alpha\vec{E}) = 0\,. \tag{35}$$

Applying the operator in the left part of Eq. (35) to the sedeon of the electromagnetic field we get

$$i\frac{1}{c}\frac{\partial\vec{H}}{\partial t} - i(\hat{\alpha}\vec{\nabla}\cdot\vec{H}) - i[\hat{\alpha}\vec{\nabla}\times\vec{H}] - \alpha\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + (\hat{\alpha}\vec{\nabla}\cdot\alpha\vec{E}) + [\hat{\alpha}\vec{\nabla}\times\alpha\vec{E}] = 0\,. \tag{36}$$

Separating values of different types in (36) we obtain the system of Maxwell equations in sedeonic form

$$\begin{aligned} &(\hat{\alpha}\vec{\nabla}\cdot\alpha\vec{E}) = 0\,, \quad [\hat{\alpha}\vec{\nabla}\times\alpha\vec{E}] = -\frac{i}{c}\frac{\partial\vec{H}}{\partial t}\,, \\ &(\hat{\alpha}\vec{\nabla}\cdot\vec{H}) = 0\,, \quad [\hat{\alpha}\vec{\nabla}\times\vec{H}] = \frac{i}{c}\frac{\partial\alpha\vec{E}}{\partial t}\,. \end{aligned} \tag{37}$$

Since $\hat{\alpha} = \alpha$ the system (37) can be transformed to the following form:

$$\begin{aligned} &(\vec{\nabla}\cdot\vec{E}) = 0\,, \quad [\vec{\nabla}\times\vec{E}] = -\frac{i}{c}\frac{\partial\vec{H}}{\partial t}\,, \\ &(\vec{\nabla}\cdot\vec{H}) = 0\,, \quad [\vec{\nabla}\times\vec{H}] = \frac{i}{c}\frac{\partial\vec{E}}{\partial t}\,. \end{aligned} \tag{38}$$

Thus the system of Maxwell equations for the electromagnetic field in a vacuum can be formulated in terms of absolute values.

The field equations (32) can be generalized for the case of a particle in an external electromagnetic field. Using substitution (24) we can write Eq. (29) in the following operator form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi - \hat{\alpha}\vec{\nabla} + \frac{ie}{\hbar c}\hat{\alpha}\vec{A} - i\frac{mc}{\hbar}\hat{\rho}\right) \times \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi + \hat{\alpha}\vec{\nabla} - \frac{ie}{\hbar c}\hat{\alpha}\vec{A} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = 0\,. \tag{39}$$

This equation enables the introduction of sedeon-scalar $\mathbf{G_0}$ and sedeon-vector $\vec{\mathbf{G}}$ fields

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi + \hat{\alpha}\vec{\nabla} - \frac{ie}{\hbar c}\hat{\alpha}\vec{A} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = \mathbf{G}_0 - \vec{\mathbf{G}}\,, \tag{40}$$

or in expanded form

$$\mathbf{G}_0 = \frac{1}{c}\frac{\partial \mathbf{W}_0}{\partial t} + \hat{\alpha}(\vec{\nabla}\cdot\vec{\mathbf{W}}) + i\frac{mc}{\hbar}\hat{\rho}\mathbf{W}_0 + \frac{ie}{\hbar c}\Phi\mathbf{W}_0 - \frac{ie}{\hbar c}\hat{\alpha}(\vec{A}\cdot\vec{\mathbf{W}})\,,$$

$$\vec{\mathbf{G}} = -\hat{\alpha}\vec{\nabla}\mathbf{W}_0 - \frac{1}{c}\frac{\partial \vec{\mathbf{W}}}{\partial t} - i\frac{mc}{\hbar}\hat{\rho}\vec{\mathbf{W}} - \hat{\alpha}[\vec{\nabla}\times\vec{\mathbf{W}}] - \frac{ie}{\hbar c}\Phi\vec{\mathbf{W}} + \frac{ie}{\hbar c}\hat{\alpha}\vec{A}\mathbf{W}_0 + \frac{ie}{\hbar c}\hat{\alpha}[\vec{A}\times\vec{\mathbf{W}}]\,.$$

Then using the field's definition we can write Eq. (39) as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi - \hat{\alpha}\vec{\nabla} + \frac{ie}{\hbar c}\hat{\alpha}\vec{A} - i\frac{mc}{\hbar}\hat{\rho}\right)(\mathbf{G}_0 - \vec{\mathbf{G}}) = 0\,.$$

This equation leads us to the following Maxwell-like first-order sedeonic equations for massive fields:

$$\begin{aligned}
(\hat{\alpha}\vec{\nabla}\cdot\vec{\mathbf{G}}) &= -\frac{1}{c}\frac{\partial \mathbf{G}_0}{\partial t} - \frac{ie}{\hbar c}\Phi\mathbf{G}_0 + \frac{ie}{\hbar c}\hat{\alpha}(\vec{A}\cdot\vec{\mathbf{G}}) + i\frac{mc}{\hbar}\hat{\rho}\mathbf{G}_0\,,\\
[\hat{\alpha}\vec{\nabla}\times\vec{\mathbf{G}}] &= \frac{1}{c}\frac{\partial \vec{\mathbf{G}}}{\partial t} + \hat{\alpha}\vec{\nabla}\mathbf{G}_0 + \frac{ie}{\hbar c}\Phi\vec{\mathbf{G}} - \frac{ie}{\hbar c}\hat{\alpha}\vec{A}\mathbf{G}_0 + \frac{ie}{\hbar c}\hat{\alpha}[\vec{A}\times\vec{\mathbf{G}}] - i\,\frac{mc}{\hbar}\hat{\rho}\vec{\mathbf{G}}\,.
\end{aligned} \tag{41}$$

5. Sedeonic First-Order Equations

As it was shown is Sec. 3 the spin interaction of a particle with external electromagnetic field can be described by the sedonic second-order equation. At that in contrast to the Dirac theory the terms describing the interaction of spin 1/2 with electric and magnetic fields are appeared in the sedeonic second-order equation as a result of sedeonic multiplication without attraction of the first-order equation. However, in sedeonic quantum mechanics we can also construct the Dirac-like first-order equations.

In this section we show that there is the special class of sedeonic wave functions, which describe particles with spin 1/2 but satisfy the first-order equations (analogous to the Dirac equation) differing in space–time transformational properties.

Let us turn to the sedeonic equation (21)

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} = -\frac{m^2c^2}{\hbar^2}\tilde{\mathbf{W}} \tag{42}$$

corresponding to the Einstein relation for energy and momentum. In this equation we can formally denote the result of action of one operator on function $\tilde{\mathbf{W}}$ as some new sedeonic function $\tilde{\mathbf{V}}$:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} = -\frac{mc}{\hbar}\tilde{\mathbf{V}}\,.$$

Then the second-order equation (42) is equivalent to the system of two first-order equations:

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} &= -\frac{mc}{\hbar}\tilde{\mathbf{V}}\,,\\ \left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{V}} &= \frac{mc}{\hbar}\tilde{\mathbf{W}}\,.\end{aligned} \tag{43}$$

Acting on the second equation of (43) by generalized operator of conjugation $\hat{\rho}$ we get

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} &= \frac{mc}{\hbar}(-\tilde{\mathbf{V}})\,,\\ \left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\hat{\rho}\tilde{\mathbf{V}} &= \frac{mc}{\hbar}\hat{\rho}\tilde{\mathbf{W}}\,.\end{aligned} \tag{44}$$

On some conditions the equations of system (44) can be absolutely equivalent. For that functions $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{W}}$ should satisfy the following relations:

$$\tilde{\mathbf{W}} = \eta\hat{\rho}\tilde{\mathbf{V}}\,, \qquad -\tilde{\mathbf{V}} = \eta\hat{\rho}\tilde{\mathbf{W}}\,,$$

where η is some constant. In particular for scalar η we obtain $\eta = \pm i$. So if the wave function $\tilde{\mathbf{W}}$ and accessory function $\tilde{\mathbf{V}}$ satisfy the condition

$$\tilde{\mathbf{V}} = \pm i\hat{\rho}\tilde{\mathbf{W}}\,, \tag{45}$$

then the wave function $\tilde{\mathbf{W}}$ satisfies the first-order equation. The sign in (45) can be chosen arbitrarily. If $\tilde{\mathbf{V}} = +i\hat{\rho}\tilde{\mathbf{W}}$ then the first-order equation has the following form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = 0\,. \tag{46}$$

Actually operator equation (46) is the set of 24 first-order equations corresponding to the different meanings of operators $\hat{\alpha}$ and $\hat{\rho}$. Using substitution (24) Eq. (46) can be generalized for the particles in an external electromagnetic field

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi + \hat{\alpha}\vec{\nabla} - \frac{ie}{\hbar c}\hat{\alpha}\vec{A} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = 0\,. \tag{47}$$

It is clear that for Eq. (46) there is also the inverse procedure of obtaining the second-order equation analogous to the procedure used in the Dirac theory. Acting on Eq. (46) by operator

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla} - i\frac{mc}{\hbar}\hat{\rho}\right),$$

we get the following equation:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla} - i\frac{mc}{\hbar}\hat{\rho}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla} + i\frac{mc}{\hbar}\hat{\rho}\right)\tilde{\mathbf{W}} = 0\,.$$

Multiplying the operators in the left part we can obtain the sedeonic second-order equation

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \hat{\alpha}\vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} = -\frac{m^2c^2}{\hbar^2}\tilde{\mathbf{W}}\,. \tag{48}$$

However, we specially emphasize though Eq. (48) coincides in form with the second-order equation (42), but the solutions to (48) should also satisfy the first-order equation (46) simultaneously. It essentially restricts the class of possible wave functions.

In conclusion to this section we note that in fact the generalized first order Dirac-like equations (46) and (47) describe particles, which do not have the fields $\mathbf{G}_0$ and $\vec{\mathbf{G}}$ (see expressions (31) and (40)).

6. Plane Wave Solutions to the First-Order Equations

Equation (46) enables the plane wave solution. Let us search a solution in the form

$$\tilde{\mathbf{W}} = \tilde{\mathbf{U}}\exp\{-i(Et - (\vec{p}\cdot\vec{r}))/\hbar\}\,,$$

where $\tilde{\mathbf{U}}$ is the wave amplitude, $\vec{p}$ is the absolute vector of momentum. Then Eq. (46) is transformed to

$$(E - c\hat{\alpha}\vec{p} - mc^2\hat{\rho})\tilde{\mathbf{U}} = 0\,. \tag{49}$$

This equation gives us the dispersion relation

$$(E^2 - p^2c^2 - m^2c^4)^8 = 0\,, \tag{50}$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$. The roots of Eq. (50) $E = \pm\sqrt{p^2c^2 + m^2c^4}$ are eighthly degenerate.

Representing in (49) $\tilde{\mathbf{U}} = \mathbf{U}_0 + \vec{\mathbf{U}}$ we obtain the following equation:

$$E\mathbf{U}_0 + E\vec{\mathbf{U}} - c\hat{\alpha}\vec{p}\mathbf{U}_0 - c\hat{\alpha}(\vec{p}\cdot\vec{\mathbf{U}}) - c\hat{\alpha}[\vec{p}\times\vec{\mathbf{U}}] - mc^2\hat{\rho}\mathbf{U}_0 - mc^2\hat{\rho}\vec{\mathbf{U}} = 0\,.$$

Separating sedeon-scalar and sedeon-vector parts we get the following system:

$$\begin{gathered} E\mathbf{U}_0 - c\hat{\alpha}(\vec{p}\cdot\vec{\mathbf{U}}) - mc^2\hat{\rho}\mathbf{U}_0 = 0\,, \\ E\vec{\mathbf{U}} - c\hat{\alpha}\vec{p}\mathbf{U}_0 - c\hat{\alpha}[\vec{p}\times\vec{\mathbf{U}}] - mc^2\hat{\rho}\vec{\mathbf{U}} = 0\,. \end{gathered} \tag{51}$$

Let momentum is directed along the Z axis, so $\vec{p} = p\mathbf{a}_3$, where p is the momentum module. Then the system (51) can be transformed in the following way:

$$E\mathbf{U}_0 - c\hat{\alpha}p\mathbf{U}_z - mc^2\hat{\rho}\mathbf{U}_0 = 0\,,$$

$$E\mathbf{U}_z - c\hat{\alpha}p\mathbf{U}_0 - mc^2\hat{\rho}\mathbf{U}_z = 0\,,$$

$$E\mathbf{U}_x + ic\hat{\alpha}p\mathbf{U}_y - mc^2\hat{\rho}\mathbf{U}_x = 0\,,$$

$$E\mathbf{U}_y - ic\hat{\alpha}p\mathbf{U}_x - mc^2\hat{\rho}\mathbf{U}_y = 0\,.$$

Thus we obtain the following relations between components of the wave function:

$$\mathbf{U}_z = \frac{\hat{\alpha}}{cp}\,(E - mc^2\hat{\rho})\mathbf{U}_0\,, \tag{52}$$

$$\mathbf{U}_y = \frac{i\hat{\alpha}}{cp}\,(E - mc^2\hat{\rho})\mathbf{U}_x\,, \tag{53}$$

or inversed relations

$$\mathbf{U}_0 = \frac{\hat{\alpha}}{cp}\,(E - mc^2\hat{\rho})\mathbf{U}_z\,,$$

$$\mathbf{U}_x = -\frac{i\hat{\alpha}}{cp}\,(E - mc^2\hat{\rho})\mathbf{U}_y\,.$$

Taking into account relations (52) and (53) we get the amplitude of wave function in the following form:

$$\tilde{\mathbf{U}} = \left(1 + \frac{\hat{\alpha}}{cp}\,(E - mc^2\hat{\rho})\mathbf{a}_3\right)\mathbf{U}_0 + \left(\mathbf{a}_1 + \frac{i\hat{\alpha}}{cp}(E - mc^2\hat{\rho})\mathbf{a}_2\right)\mathbf{U}_x\,, \tag{54}$$

where $\mathbf{U}_0$ and $\mathbf{U}_x$ are arbitrary sedeon-scalar constants. The expression (54) can also be represented in the following compact form:

$$\tilde{\mathbf{U}} = \left(1 + \frac{\hat{\alpha}}{cp}(E - mc^2\hat{\rho})\mathbf{a}_3\right)(\mathbf{U}_0 + \mathbf{U}_x\mathbf{a}_1)\,. \tag{55}$$

7. Sedeonic First-Order Equations for Massless Particles

Using the results of Sec. 5 we can indicate the generalized sedeonic first-order equation for massless particles (neutrinos):

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \hat{\alpha}\vec{\nabla}\right)\tilde{\mathbf{W}} = 0\,, \tag{56}$$

where as before the operator $\hat{\alpha}$ takes any meaning from $\hat{\alpha} \in \{\pm\hat{\mathbf{e}}_{\mathbf{r}}, \pm\hat{\mathbf{e}}_{\mathbf{t}}, \pm\hat{\mathbf{e}}_{\mathbf{rt}}\}$ and describes different transformational properties. The operator equation (56) is the set of three groups equations differing in properties of operators $\hat{\alpha}$.

Equation (56) enables the plane wave solution. Let us search a solution in the form

$$\tilde{\mathbf{W}} = \tilde{\mathbf{U}}\exp\{-i(Et - (\vec{p}\cdot\vec{r}))/\hbar\}\,,$$

then Eq. (56) is transformed to

$$(E - c\hat{\alpha}\vec{p})\tilde{\mathbf{U}} = 0\,.$$

The dispersion relation for Eq. (56) has the form

$$E = \gamma_\nu cp\,,$$

where p is the momentum module, $\gamma_\nu = +1$ for neutrino and $\gamma_\nu = -1$ for antineutrino.

Let momentum is directed along the Z axis, so $\vec{p} = p\mathbf{a}_3$. Then the amplitude of the wave function can be obtained directly from the expression (55) if we take the mass equal to zero:

$$\tilde{\mathbf{U}} = (1 + \gamma_\nu \hat{\alpha}\mathbf{a}_3)(\mathbf{U}_0 + \mathbf{U}_x \mathbf{a}_1)\,,$$

where $\mathbf{U}_0$ and $\mathbf{U}_x$ are arbitrary sedeon-scalar complex constants. Then the generalized plane wave solution to Eq. (56) can be written in the following form:

$$\tilde{\mathbf{W}} = (1 + \gamma_\nu \hat{\alpha}\mathbf{a}_3)(\mathbf{U}_0 + \mathbf{U}_x \mathbf{a}_1)\exp\{ip(z - \gamma_\nu ct)/\hbar\}\,.$$

Concluding this section we would like to indicate one special sedeonic first-order equation for the massless particle corresponding to special case $\hat{\alpha} = \pm 1$. In this case Eq. (56) takes the form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\tilde{\mathbf{W}} = 0\,, \tag{57}$$

or adjoint form

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right)\tilde{\mathbf{W}} = 0\,. \tag{58}$$

Note that the operators in the left part of Eqs. (57) and (58) are not changed under space or time conjugation at all. We also emphasize that these equations do not correspond to any first-order equations for the massive particles from the system of (46), so it is the separate case. Equation (57) has the following plane wave solution for particle (ν):

$$\tilde{\mathbf{W}}_\nu = (1 + \mathbf{a}_3)(\mathbf{U}_0 + \mathbf{U}_x \mathbf{a}_1)\exp\{ip(z - ct)/\hbar\} \tag{59}$$

and for the antiparticle ($\bar{\nu}$):

$$\tilde{\mathbf{W}}_{\bar{\nu}} = (1 - \mathbf{a}_3)(\mathbf{U}_0 + \mathbf{U}_x \mathbf{a}_1)\exp\{ip(z + ct)/\hbar\}\,. \tag{60}$$

Consequently Eq. (57) describes simultaneously the particle and the antiparticle. It is clearly seen from (59) and (60) that wave functions of massless particle and antiparticle are the eigenfunctions of the operator $\hat{\mathbf{a}}_3$ (see Ref. 25). At that the wave function of particle corresponds to the state with the eigenvalue $\lambda = +1$ and the wave function of antiparticle corresponds to the state with the eigenvalue $\lambda = -1$. So the expressions (59) and (60) describe polarized particles. Contrary to (57) Eq. (58) describes the particle in the state with the eigenvalue $\lambda = -1$ and antiparticle in the state with the eigenvalue $\lambda = +1$.

8. Conclusion

Thus in this paper we represented sixteen-component values "sedeons," generating associative noncommutative algebra. The sedeon is the complicated space–time object consisting of absolute scalar, space scalar, time scalar, space–time scalar, absolute vector, space vector, time vector, and space–time vector. All these values are differed with respect to spatial and time inversion.

We proposed a scheme for constructing relativistic quantum mechanics using sedeonic space–time operators and sedeonic wave functions. It was shown that the sedeonic second-order equation, corresponding to the Einstein relation between energy and momentum, correctly describes the interaction between spin 1/2 and the electromagnetic field. It is established that the sedeonic wave function of a particle in the state with defined spin projection has the specific space–time structure in the form of a sedeonic oscillator with two spatial polarizations: longitudinal linear and transverse circular.

We showed that the sedeonic second-order wave equation can be reformulated in the form of the system of the first-order Maxwell-like equations for the massive fields. We proposed the set of sedeonic first-order equations analogous to the Dirac equation. It was shown that the sedeonic Dirac-like first-order equations describe particles, which do not have massive fields. In dependence of space–time operators these equations describe three different kinds of particles (leptons) which are differed by space–time transformational properties. We proposed three kinds of sedeonic first-order equations for massless particles (neutrinos) corresponding to the three types of equations for massive leptons. Besides we indicated the special type of absolute equation for massless particle, which does not correspond to any leptonic equation at all.

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