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Octonic representation of electromagnetic field equations

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In this paper we represent eight-component values "octons," generating associative noncommutative algebra. It is shown that the electromagnetic field in a vacuum can be described by a generalized octonic equation, which leads both to the wave equations for potentials and fields and to the system of Maxwell's equations. The octonic algebra allows one to perform compact combined calculations simultaneously with scalars, vectors, pseudoscalars, and pseudovectors. Examples of such calculations are demonstrated by deriving the relations for energy, momentum, and Lorentz invariants of the electromagnetic field. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3041499]

I. INTRODUCTION

Hypercomplex numbers[1–4] especially quaternions are widely used in relativistic mechanics, electrodynamics, quantum mechanics, and quantum field theory[3–10] (see also the bibliographical review Ref. 11). The structure of quaternions with four components (scalar and vector) corresponds to the relativistic space-time structure, which allows one to realize the quaternionic generalization of quantum mechanics.[5–8] However, quaternions do not include pseudoscalar and pseudovector components. Therefore for describing all types of physical values the eight-component hypercomplex numbers enclosing scalars, vectors, pseudoscalars, and pseudovectors are more appropriate.

The idea of applying the eight-component hypercomplex numbers for the description of the electromagnetic field is quite natural since Maxwell's equations are the system of four equations for scalar, vector, pseudoscalar, and pseudovector values. There are a lot of papers that describe the attempts to realize representations of Maxwell equations using different eight-component hypercomplex numbers such as biquaternions,[4,12,13] octonions,[14–16] and multivectors generating the associative Clifford algebras.[17–19] However, all considered systems of hypercomplex numbers do not have a consistent vector interpretation, which leads to difficulties in the description of vectorial electromagnetic fields.

This paper is devoted to describing electromagnetic fields on the basis of eight-component values "octons," which generate associative noncommutative algebra and have the clearly defined simple geometric sense. The paper has the following structure. In Sec. II we consider the peculiarities of the eight-component octonic algebra. In Sec. III the generalized octonic equations for the electromagnetic field in a vacuum are formulated. In Sec. IV the derivations of the relations for energy, momentum, and Lorentz invariants of electromagnetic field are demonstrated.

II. ALGEBRA OF OCTONS

The values of four types (scalars, vectors, pseudoscalars, and pseudovectors) differing with respect to spatial inversion are used for the description of the electromagnetic field. All these values can be integrated into one spatial object. For this purpose in the present paper we propose the special eight-component values, which will be named octons for short.

The eight-component octon $\breve{G}$ is defined by the following expression:

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TABLE I. The rules of multiplication and commutation for the octon's unit vectors.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{a}_0$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$	$\mathbf{a}_1$	$\mathbf{a}_0$	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$	$\mathbf{a}_2$	$-i\mathbf{a}_3$	$\mathbf{a}_0$	$i\mathbf{a}_1$
$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1	$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	$\mathbf{a}_0$
$\mathbf{a}_0$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{a}_1$	$\mathbf{a}_0$	$i\mathbf{a}_3$	$-i\mathbf{a}_2$	$\mathbf{e}_1$	1	$i\mathbf{e}_3$	$-i\mathbf{e}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	$\mathbf{a}_0$	$i\mathbf{a}_1$	$\mathbf{e}_2$	$-i\mathbf{e}_3$	1	$i\mathbf{e}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	$\mathbf{a}_0$	$\mathbf{e}_3$	$i\mathbf{e}_2$	$-i\mathbf{e}_1$	1

$$\breve{G} = c_0\mathbf{e_0} + c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} + d_0\mathbf{a_0} + d_1\mathbf{a_1} + d_2\mathbf{a_2} + d_3\mathbf{a_3}, \tag{1}$$

where $\mathbf{e_0} \equiv 1$, values $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ are axial unit vectors (pseudovectors), $\mathbf{a_0}$ is the pseudoscalar unit, $\mathbf{a_1}$, $\mathbf{a_2}$, and $\mathbf{a_3}$ are polar unit vectors. The octonic components c_n and d_n $(n=0,1,2,3)$ are numbers (complex, in general). Thus the octon is the sum of a scalar, pseudovector, pseudoscalar, and vector. The full octon basis is

$$\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{a_0}, \mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}. \tag{2}$$

The rules for multiplication of polar and axial basis vectors are formulated taking into account the symmetry of their products with respect to the operation of spatial inversion. For polar unit vectors $\mathbf{a}_k$ $(k=1,2,3)$ the following rules of multiplication take place:

$$\mathbf{a}_k^2 = 1, \tag{3}$$

$$\mathbf{a}_j\mathbf{a}_k = -\mathbf{a}_k\mathbf{a}_j \quad (\text{for } j \neq k, \quad j = 1,2,3). \tag{4}$$

The conditions (4) describe the property of noncommutativity for vector product. The same rules are defined for axial unit vector multiplication,

$$\mathbf{e}_k^2 = 1, \tag{5}$$

$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j \quad (\text{for } j \neq k). \tag{6}$$

The rules (3)–(6) allow one to represent the length square of any polar or axial vector as the sum of squares of its components. We emphasize that the square of vector length is positively defined. The rules (3) and (5) lead to some special requirements for the vector product in octonic algebra. Let $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ be the right Cartesian bases and corresponding unit vectors are parallel to each other. Taking into consideration (3)–(5) and the fact that the product of two different polar vectors is an axial vector, we can represent the rules for cross multiplication of polar unit vectors in the following way:

$$\mathbf{a_1}\mathbf{a_2} = i\mathbf{e_3}, \quad \mathbf{a_2}\mathbf{a_3} = i\mathbf{e_1}, \quad \mathbf{a_3}\mathbf{a_1} = i\mathbf{e_2}, \tag{7}$$

where i is the imaginary unit. Then the rules of multiplication for axial basis units can be written

$$\mathbf{e_1}\mathbf{e_2} = i\mathbf{e_3}, \quad \mathbf{e_2}\mathbf{e_3} = i\mathbf{e_1}, \quad \mathbf{e_3}\mathbf{e_1} = i\mathbf{e_2}. \tag{8}$$

Let us define the pseudoscalar unit $\mathbf{a_0}$ as the product of parallel unit vectors corresponding to the different bases,

$$\mathbf{a_0} = \mathbf{a}_k\mathbf{e}_k. \tag{9}$$

Squaring (9) we can see that $\mathbf{a_0}^2=1$. Note that the unit $\mathbf{a_0}$ commutates with each unit vector.

Summarized commutation and multiplication rules are represented in Table I.

We would like to emphasize especially that octonic algebra is associative. The property of associativity follows directly from multiplication rules.

Thus the octon $\breve{G}$ (1) is the sum of the scalar value c_0, the pseudovector value (axial vector) $\overleftrightarrow{c}=c_1\mathbf{e_1}+c_2\mathbf{e_2}+c_3\mathbf{e_3}$, the pseudoscalar value $\tilde{d}_0=d_0\mathbf{a_0}$, and the vector value (polar vector) $\vec{d}=d_1\mathbf{a_1}+d_2\mathbf{a_2}+d_3\mathbf{a_3}$,

$$\breve{G}=c_0+\overleftrightarrow{c}+\tilde{d}_0+\vec{d}.$$

Hereinafter octons will be indicated by the "⌣" symbol, pseudovectors by a double arrow "↔," pseudoscalars by a wave "∼," and vectors by an arrow "→." The values c_k and d_k ($k=1,2,3$) are the projections of the axial vector $\overleftrightarrow{c}$ and the polar vector $\vec{d}$ on the corresponding unit vector directions. Note, that equality of two octons means the equality of all corresponding components.

Let us consider the rules of multiplication of two octons in detail. First, the result of octonic multiplication of two polar vectors $\vec{d}_1$ and $\vec{d}_2$ is the sum of scalar and pseudovector values,

$$\begin{aligned}\vec{d}_1\vec{d}_2&=\{d_{11}\mathbf{a_1}+d_{12}\mathbf{a_2}+d_{13}\mathbf{a_3}\}\{d_{21}\mathbf{a_1}+d_{22}\mathbf{a_2}+d_{23}\mathbf{a_3}\}=\{d_{11}d_{21}+d_{12}d_{22}+d_{13}d_{23}\}+i\{d_{12}d_{23}\\&\quad-d_{13}d_{22}\}\mathbf{e_1}+i\{d_{13}d_{21}-d_{11}d_{23}\}\mathbf{e_2}+i\{d_{11}d_{22}-d_{12}d_{21}\}\mathbf{e_3}.\end{aligned} \tag{10}$$

Hereinafter we will denote the scalar multiplication (internal product) by the symbol "·" and round brackets,

$$(\overleftrightarrow{c}_1\cdot\overleftrightarrow{c}_2)=c_{11}c_{21}+c_{12}c_{22}+c_{13}c_{23},$$

$$(\vec{d}_1\cdot\vec{d}_2)=d_{11}d_{21}+d_{12}d_{22}+d_{13}d_{23},$$

$$(\overleftrightarrow{c}\cdot\vec{d})=\{c_1d_1+c_2d_2+c_3d_3\}\mathbf{a_0}.$$

Vector multiplication (external product) will be denoted by the symbol "×" and square brackets,

$$[\overleftrightarrow{c}_1\times\overleftrightarrow{c}_2]=i\{c_{12}c_{23}-c_{13}c_{22}\}\mathbf{e_1}+i\{c_{13}c_{21}-c_{11}c_{23}\}\mathbf{e_2}+i\{c_{11}c_{22}-c_{12}c_{21}\}\mathbf{e_3},$$

$$[\vec{d}_1\times\vec{d}_2]=i\{d_{12}d_{23}-d_{13}d_{22}\}\mathbf{e_1}+i\{d_{13}d_{21}-d_{11}d_{23}\}\mathbf{e_2}+i\{d_{11}d_{22}-d_{12}d_{21}\}\mathbf{e_3},$$

$$[\overleftrightarrow{c}\times\vec{d}]=i\{c_2d_3-c_3d_2\}\mathbf{a_1}+i\{c_3d_1-c_1d_3\}\mathbf{a_2}+i\{c_1d_2-c_2d_1\}\mathbf{a_3}.$$

In all other cases round and square brackets will be used for the priority definition. Thus taking into account the considered designations, the octonic product of two vectors (10) can be represented as the sum of scalar and vector products,

$$\vec{d}_1\vec{d}_2=(\vec{d}_1\cdot\vec{d}_2)+[\vec{d}_1\times\vec{d}_2].$$

Then the product of two octons can be represented in the following form:

$$\begin{aligned}\breve{G}_1\breve{G}_2&=\{c_{10}+\overleftrightarrow{c}_1+\tilde{d}_{10}+\vec{d}_1\}\{c_{20}+\overleftrightarrow{c}_2+\tilde{d}_{20}+\vec{d}_2\}=c_{10}c_{20}+c_{10}\overleftrightarrow{c}_2+c_{10}\tilde{d}_{20}+c_{10}\vec{d}_2+c_{20}\overleftrightarrow{c}_1+(\overleftrightarrow{c}_1\cdot\overleftrightarrow{c}_2)\\&\quad+[\overleftrightarrow{c}_1\times\overleftrightarrow{c}_2]+\tilde{d}_{20}\overleftrightarrow{c}_1+(\overleftrightarrow{c}_1\cdot\vec{d}_2)+[\overleftrightarrow{c}_1\times\vec{d}_2]+\tilde{d}_{10}c_{20}+\tilde{d}_{10}\overleftrightarrow{c}_2+\tilde{d}_{10}\tilde{d}_{20}+\tilde{d}_{10}\vec{d}_2+c_{20}\vec{d}_1+(\vec{d}_1\cdot\overleftrightarrow{c}_2)\\&\quad+[\vec{d}_1\times\overleftrightarrow{c}_2]+\tilde{d}_{20}\vec{d}_1+(\vec{d}_1\cdot\vec{d}_2)+[\vec{d}_1\times\vec{d}_2].\end{aligned}$$

Some correspondence between the algebra of octons and Gibbs vector algebra was discussed in Ref. 20. We can also indicate some connection between octonic algebra and algebra of quaternions. Indeed it is easy to see that the rules of octonic multiplication and commutation take place for the values based on the quaternionic imaginary units q_k ($k=1,2,3$; $q_k^2=-1$),

$$\mathbf{e}_k = iq_k, \quad \mathbf{a}_k = iq_k\mathbf{a_0},$$

but it needs the introduction of a new (nonquaternionic) pseudoscalar element $\mathbf{a_0}$.

In conclusion in this section we would like to note that formally the algebra of octons can be considered as the variant of complexified Clifford algebra. However, in contrast to Clifford algebra the octonic unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are the real true vectors but not complex numbers. In this connection octons have a clear well-defined space-geometry sense.

III. OCTONIC FORM OF ELECTRODYNAMICS EQUATIONS

The octonic algebra can be naturally applied to the description of the electromagnetic field in a vacuum. The potential of the electromagnetic field is represented as an incomplete four-component octon,

$$\breve{\Pi} = \varphi + \vec{A} = \varphi + A_1\mathbf{a_1} + A_2\mathbf{a_2} + A_3\mathbf{a_3},$$

where φ is the scalar potential and $\vec{A}$ is the vector potential. The four-component current also can be defined as an incomplete octon,

$$\breve{J} = 4\pi\rho + \frac{4\pi}{c}\vec{j} = 4\pi\rho + \frac{4\pi}{c}(j_1\mathbf{a_1} + j_2\mathbf{a_2} + j_3\mathbf{a_3}).$$

Then using the octonic differentiation operator

$$\hat{P} = \left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\mathbf{a_1} + \frac{\partial}{\partial x_2}\mathbf{a_2} + \frac{\partial}{\partial x_3}\mathbf{a_3}\right)$$

and conjugated operator

$$\hat{P}^{+} = \left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x_1}\mathbf{a_1} - \frac{\partial}{\partial x_2}\mathbf{a_2} - \frac{\partial}{\partial x_3}\mathbf{a_3}\right),$$

we can write the generalized equation of electrodynamics in the compact octonic form,

$$\hat{P}^{+}\hat{P}\breve{\Pi} = \breve{J}. \tag{11}$$

Indeed multiplying $\hat{P}^{+}$ and $\hat{P}$ operators in (11), we obtain the wave equation for potentials of electromagnetic field in the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta - [\vec{\nabla} \times \vec{\nabla}]\right)\breve{\Pi} = \breve{J}. \tag{12}$$

For the potentials described by twice differentiable functions $[\vec{\nabla}\times\vec{\nabla}]\breve{\Pi}=0$, Eq. (12) becomes

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\breve{\Pi} = \breve{J}. \tag{13}$$

Separating scalar and vector parts in (13), we obtain ordinary wave equations for the scalar and vector potentials,

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 4\pi\rho,$$

$$\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}-\Delta\vec{A}=\frac{4\pi}{c}\vec{j}.$$

On the other hand, applying in Eq. (11) operators $\hat{P}$ and $\hat{P}^{+}$ one after another to the octonic potential $\breve{\Pi}$, we can obtain first

$$\hat{P}\breve{\Pi}=\left(\frac{1}{c}\frac{\partial}{\partial t}+\vec{\nabla}\right)(\varphi+\vec{A})=\frac{1}{c}\frac{\partial\varphi}{\partial t}+\vec{\nabla}\varphi+\frac{1}{c}\frac{\partial\vec{A}}{\partial t}+(\vec{\nabla}\cdot\vec{A})+[\vec{\nabla}\times\vec{A}]. \tag{14}$$

We will use the standard definitions of the electric and magnetic fields in octonic form,

$$\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi, \quad \overleftrightarrow{H}=-i[\vec{\nabla}\times\vec{A}].$$

Taking into account Lorentz gauge condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}+(\vec{\nabla}\cdot\vec{A})=0,$$

we can rewrite the result of $\hat{P}$ operation in (14) as

$$\hat{P}\breve{\Pi}=-\vec{E}+i\overleftrightarrow{H}, \tag{15}$$

where in the right part of (15) the octon of electromagnetic field $\breve{F}$ is written

$$\breve{F}=-\vec{E}+i\overleftrightarrow{H}.$$

Consequently Eq. (11) becomes

$$P^{+}\breve{F}=\breve{J}. \tag{16}$$

Applying the operator $\hat{P}^{+}$ to the octon of the electromagnetic field $\breve{F}$, we get

$$\frac{i}{c}\frac{\partial\overleftrightarrow{H}}{\partial t}-i(\vec{\nabla}\cdot\overleftrightarrow{H})-i[\vec{\nabla}\times\overleftrightarrow{H}]-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+(\vec{\nabla}\cdot\vec{E})+[\vec{\nabla}\times\vec{E}]=4\pi\rho+\frac{4\pi}{c}\vec{j}. \tag{17}$$

Separating scalar, vector, pseudoscalar, and pseudovector terms in Eq. (17), we get the system of Maxwell equations in octonic form

$$(\vec{\nabla}\cdot\vec{E})=4\pi\rho \quad \text{(scalar term)},$$

$$[\vec{\nabla}\times\vec{E}]=-\frac{i}{c}\frac{\partial\overleftrightarrow{H}}{\partial t} \quad \text{(pseudovector term)},$$

$$(\vec{\nabla}\cdot\overleftrightarrow{H})=0 \quad \text{(pseudoscalar term)},$$

$$[\vec{\nabla}\times\overleftrightarrow{H}]=\frac{4\pi i}{c}\vec{j}+\frac{i}{c}\frac{\partial\vec{E}}{\partial t} \quad \text{(vector term)}. \tag{18}$$

The system (18) coincides with Maxwell equations.

Applying operator $\hat{P}$ to both parts of Eq. (16), one can obtain the wave equation for $\vec{E}$ and $\overleftrightarrow{H}$ fields,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\Delta\right)(i\overleftrightarrow{H}-\vec{E}) = \frac{4\pi}{c}\frac{\partial\rho}{\partial t}+4\pi\vec{\nabla}\rho+\frac{4\pi}{c^2}\frac{\partial\vec{j}}{\partial t}+\frac{4\pi}{c}(\vec{\nabla}\cdot\vec{j})+\frac{4\pi}{c}[\vec{\nabla}\times\vec{j}].$$

Separating scalar, vector, pseudoscalar, and pseudovector terms, we obtain the system of three equations,

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}+\Delta\vec{E} = -4\pi\vec{\nabla}\rho-\frac{4\pi}{c^2}\frac{\partial\vec{j}}{\partial t},$$

$$\frac{1}{c^2}\frac{\partial^2\overleftrightarrow{H}}{\partial t^2}+\Delta\overleftrightarrow{H} = -\frac{4\pi}{c}i[\vec{\nabla}\times\vec{j}],$$

$$\frac{\partial\rho}{\partial t}+(\vec{\nabla}\cdot\vec{j}) = 0. \tag{19}$$

The first two equations in (19) are the wave equations for electric and magnetic fields and the third one is the continuity equation.

IV. RELATIONS FOR ENERGY, MOMENTUM, AND LORENTZ INVARIANTS OF ELECTROMAGNETIC FIELD

The octonic algebra allows one to provide the combined calculus with different types of values simultaneously. For example, in this section we obtain the relations for energy, momentum, and Lorentz invariants of electromagnetic field.

Multiplying both parts of expression (17) on octon $(\vec{E}+i\overleftrightarrow{H})$ from the left, we can obtain the following octonic equation:

$$(\vec{E}+i\overleftrightarrow{H})\left(\frac{i}{c}\frac{\partial\overleftrightarrow{H}}{\partial t}-i(\vec{\nabla}\cdot\overleftrightarrow{H})-i[\vec{\nabla}\times\overleftrightarrow{H}]-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+(\vec{\nabla}\cdot\vec{E})+[\vec{\nabla}\times\vec{E}]\right) = (\vec{E}+i\overleftrightarrow{H})\left(4\pi\rho+\frac{4\pi}{c}\vec{j}\right).$$

After multiplication we get

$$\begin{aligned}
&\frac{i}{c}\left(\vec{E}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)+\frac{i}{c}\left[\vec{E}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-\frac{1}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-\frac{1}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-i\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})+\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})-i(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])\\
&\quad -i[\vec{E}\times[\vec{\nabla}\times\overleftrightarrow{H}]]+(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+[\overleftrightarrow{H}\times[\vec{\nabla}\times\overleftrightarrow{H}]]-\frac{1}{c}\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)-\frac{1}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right]\\
&\quad -\frac{i}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\vec{E}}{\partial t}\right)-\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\vec{E}}{\partial t}\right]+\vec{E}(\vec{\nabla}\cdot\vec{E})+i\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})+(\vec{E}\cdot[\vec{\nabla}\times\vec{E}])+[\vec{E}\times[\vec{\nabla}\times\vec{E}]]\\
&\quad +i(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}])+i[\overleftrightarrow{H}\times[\vec{\nabla}\times\vec{E}]] = 4\pi\rho\vec{E}+i4\pi\rho\overleftrightarrow{H}+\frac{4\pi}{c}(\vec{j}\cdot\vec{E})-\frac{4\pi}{c}[\vec{j}\times\vec{E}]+i\frac{4\pi}{c}(\vec{j}\cdot\overleftrightarrow{H})\\
&\quad -i\frac{4\pi}{c}[\vec{j}\times\overleftrightarrow{H}].
\end{aligned} \tag{20}$$

Separating in (20) values of different types (scalar, vector, pseudoscalar, and pseudovector) we obtain four relations. The scalar part of Eq. (20) is written

$$-\frac{1}{c}\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)-\frac{1}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-i(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+i(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}]) = \frac{4\pi}{c}(\vec{j}\cdot\vec{E}).$$

Taking into account

$$\left(\overleftrightarrow{H}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)=\frac{1}{2}\frac{\partial}{\partial t}(\overleftrightarrow{H}\cdot\overleftrightarrow{H})=\frac{1}{2}\frac{\partial}{\partial t}\overleftrightarrow{H}^2,$$

$$\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)=\frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E})=\frac{1}{2}\frac{\partial}{\partial t}\vec{E}^2,$$

as well as

$$(\vec{\nabla}\cdot[\vec{E}\times\overleftrightarrow{H}])=(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}])-(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}]),$$

we obtain the following expression:

$$\frac{\partial}{\partial t}\left(\frac{\vec{E}^2+\overleftrightarrow{H}^2}{8\pi}\right)-\frac{c}{4\pi}i(\vec{\nabla}\cdot[\vec{E}\times\overleftrightarrow{H}])+(\vec{j}\cdot\vec{E})=0,$$

which is the well known Poynting theorem.

The pseudoscalar part of Eq. (20) is

$$\frac{i}{c}\left(\vec{E}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-\frac{i}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\vec{E}}{\partial t}\right)+(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+(\vec{E}\cdot[\vec{\nabla}\times\vec{E}])=\frac{4\pi}{c}i(\vec{j}\cdot\overleftrightarrow{H}). \tag{21}$$

The expression (21) is the trivial corollary, which follows from vector and pseudovector Maxwell equations (18).

The vector part of Eq. (20) is

$$\begin{aligned}&\frac{i}{c}\left[\vec{E}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\vec{E}}{\partial t}\right]+\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})+[\overleftrightarrow{H}\times[\vec{\nabla}\times\overleftrightarrow{H}]]+\vec{E}(\vec{\nabla}\cdot\vec{E})+[\vec{E}\times[\vec{\nabla}\times\vec{E}]]\\&\quad=4\pi\rho\vec{E}-\frac{4\pi}{c}i[\vec{j}\times\overleftrightarrow{H}].\end{aligned}\tag{22}$$

From (22) we obtain a well known relation between energy and momentum of the electromagnetic field,

$$\vec{\nabla}\left(\frac{\vec{E}^2+\overleftrightarrow{H}^2}{8\pi}\right)-\frac{i}{4\pi c}\frac{\partial}{\partial t}[\vec{E}\times\overleftrightarrow{H}]+\rho\vec{E}-\frac{i}{c}[\vec{j}\times\overleftrightarrow{H}]=\frac{1}{4\pi}\{\vec{E}(\vec{\nabla}\cdot\vec{E})+(\vec{E}\cdot\vec{\nabla})\vec{E}+\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})+(\overleftrightarrow{H}\cdot\vec{\nabla})\overleftrightarrow{H}\}.$$

Finally, the pseudovector part of (20) is

$$\begin{aligned}&-\frac{1}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-\frac{1}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right]+i\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})-i\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})+i[\overleftrightarrow{H}\times[\vec{\nabla}\times\vec{E}]]-i[\vec{E}\times[\vec{\nabla}\times\overleftrightarrow{H}]]\\&\quad=4\pi i\rho\overleftrightarrow{H}-\frac{4\pi}{c}[\vec{j}\times\vec{E}].\end{aligned}$$

After simple manipulations we obtain the following relation:

$$\begin{aligned}&[\overleftrightarrow{H}\times[\vec{\nabla}\times\vec{E}]]-[\vec{E}\times[\vec{\nabla}\times\overleftrightarrow{H}]]+\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})-\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})+\frac{i}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right]+\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]\\&\quad=4\pi\rho\overleftrightarrow{H}+\frac{4\pi}{c}i[\vec{j}\times\vec{E}].\end{aligned}$$

Thus in octonic algebra the simple procedure of multiplication of Eq. (17) on the electromagnetic field octon allows one to obtain simultaneously all the well known relations for the energy and momentum of the electromagnetic field.

On the other hand if we multiply Eq. (17) on the octon $(i\overleftrightarrow{H}-\vec{E})$, we get

$$(i\overleftrightarrow{H}-\vec{E})\left(\frac{i}{c}\frac{\partial\overleftrightarrow{H}}{\partial t}-i(\vec{\nabla}\cdot\overleftrightarrow{H})-i[\vec{\nabla}\times\overleftrightarrow{H}]-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+(\vec{\nabla}\cdot\vec{E})+[\vec{\nabla}\times\vec{E}]\right)=(i\overleftrightarrow{H}-\vec{E})\left(4\pi\rho+\frac{4\pi}{c}\vec{j}\right).$$

Performing multiplication we obtain

$$\begin{aligned}&-\frac{1}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-\frac{1}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-\frac{i}{c}\left(\vec{E}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-\frac{i}{c}\left[\vec{E}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]+\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})+i\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})\\&+(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+[\overleftrightarrow{H}\times[\vec{\nabla}\times\overleftrightarrow{H}]]+i(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+i[\vec{E}\times[\vec{\nabla}\times\overleftrightarrow{H}]]-\frac{i}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\vec{E}}{\partial t}\right)\\&-\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\vec{E}}{\partial t}\right]+\frac{1}{c}\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)+\frac{1}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right]+i\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})-\vec{E}(\vec{\nabla}\cdot\vec{E})+i(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}])\\&+i[\overleftrightarrow{H}\times[\vec{\nabla}\times\vec{E}]]-(\vec{E}\cdot[\vec{\nabla}\times\vec{E}])-[\vec{E}\times[\vec{\nabla}\times\vec{E}]]\\&=i4\pi\rho\overleftrightarrow{H}-4\pi\rho\vec{E}+i\frac{4\pi}{c}(\vec{j}\cdot\overleftrightarrow{H})-i\frac{4\pi}{c}[\vec{j}\times\overleftrightarrow{H}]-\frac{4\pi}{c}(\vec{j}\cdot\vec{E})+\frac{4\pi}{c}[\vec{j}\times\vec{E}].\end{aligned}\tag{23}$$

The scalar part of Eq. (23) is written as

$$-\frac{1}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)+\frac{1}{c}\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right)+i(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+i(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}])=-\frac{4\pi}{c}(\vec{j}\cdot\vec{E}).$$

This expression leads to the relation for the Lorentz invariant $\vec{E}^2-\overleftrightarrow{H}^2$, which can be represented as

$$\frac{\partial}{\partial t}\left(\frac{\vec{E}^2-\overleftrightarrow{H}^2}{8\pi}\right)+\frac{c}{4\pi}i\{(\vec{E}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])+(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\vec{E}])\}+(\vec{j}\cdot\vec{E})=0.$$

The pseudoscalar part of Eq. (23) is written as

$$-\frac{i}{c}\left(\vec{E}\cdot\frac{\partial\overleftrightarrow{H}}{\partial t}\right)-\frac{i}{c}\left(\overleftrightarrow{H}\cdot\frac{\partial\vec{E}}{\partial t}\right)+(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])-(\vec{E}\cdot[\vec{\nabla}\times\vec{E}])=i\frac{4\pi}{c}(\vec{j}\cdot\overleftrightarrow{H}),$$

which leads to

$$\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\cdot\overleftrightarrow{H})+i(\overleftrightarrow{H}\cdot[\vec{\nabla}\times\overleftrightarrow{H}])-i(\vec{E}\cdot[\vec{\nabla}\times\vec{E}])+\frac{4\pi}{c}(\vec{j}\cdot\overleftrightarrow{H})=0.\tag{24}$$

The expression (24) is the relation for the second Lorentz invariant $(\vec{E}\cdot\overleftrightarrow{H})$.

The vector part of octonic equation (23) is written as

$$\begin{aligned}&-\frac{i}{c}\left[\vec{E}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]-\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\vec{E}}{\partial t}\right]+\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})-\vec{E}(\vec{\nabla}\cdot\vec{E})+[\overleftrightarrow{H}\times[\vec{\nabla}\times\overleftrightarrow{H}]]-[\vec{E}\times[\vec{\nabla}\times\vec{E}]]\\&=-4\pi\rho\vec{E}-i\frac{4\pi}{c}[\vec{j}\times\overleftrightarrow{H}].\end{aligned}$$

After transformation we obtain the following expression:

$$\vec{\nabla}\left(\frac{\vec{E}^2-\overleftrightarrow{H}^2}{8\pi}\right)-\frac{i}{4\pi c}\left\{\left[\vec{E}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]+\left[\overleftrightarrow{H}\times\frac{\partial\vec{E}}{\partial t}\right]\right\}+\rho\vec{E}+\frac{i}{c}[\vec{j}\times\overleftrightarrow{H}]$$
$$=\frac{1}{4\pi}\{\vec{E}(\vec{\nabla}\cdot\vec{E})+(\vec{E}\cdot\vec{\nabla})\vec{E}-\overleftrightarrow{H}(\vec{\nabla}\cdot\overleftrightarrow{H})-(\overleftrightarrow{H}\cdot\vec{\nabla})\overleftrightarrow{H}\}.$$

Finally, the pseudovector part of Eq. (23) is given by

$$-\frac{1}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]+\frac{1}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right]+i\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})+i\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})+i[\overleftrightarrow{H}\times[\vec{\nabla}\times\vec{E}]]+i[\vec{E}\times[\vec{\nabla}\times\overleftrightarrow{H}]]$$
$$=4\pi i\rho\overleftrightarrow{H}+\frac{4\pi}{c}[\vec{j}\times\vec{E}].$$

After conversion we obtain

$$\vec{\nabla}(\vec{E}\cdot\overleftrightarrow{H})=\overleftrightarrow{H}(\vec{\nabla}\cdot\vec{E})+\vec{E}(\vec{\nabla}\cdot\overleftrightarrow{H})+(\vec{E}\cdot\vec{\nabla})\overleftrightarrow{H}+(\overleftrightarrow{H}\cdot\vec{\nabla})\vec{E}-4\pi\rho\overleftrightarrow{H}+\frac{4\pi}{c}i[\vec{j}\times\vec{E}]+\frac{i}{c}\left[\overleftrightarrow{H}\times\frac{\partial\overleftrightarrow{H}}{\partial t}\right]$$
$$-\frac{i}{c}\left[\vec{E}\times\frac{\partial\vec{E}}{\partial t}\right],$$

which is the expression for the gradient of the second Lorentz invariant.

V. CONCLUSION

Thus we have represented the eight-component octons (enclosing scalar, pseudovector, pseudoscalar, and vector values) generating noncommutative associative algebra. On the basis of octonic algebra the generalized octonic equation for the electromagnetic field has been proposed. It was shown that this equation leads both to the wave equations for potentials and fields and to the system of Maxwell equations.

Octonic calculus methods have been applied to the derivation of the relations for energy, momentum, and Lorentz invariants of electromagnetic field. It was shown that in octonic algebra the complicated relations between values characterizing the electromagnetic field are obtained as a result of simple octonic multiplication.

The proposed octonic algebra is also convenient and natural for the generalization of the relativistic quantum mechanics equations on the basis of octonic wave functions and octonic operators that will be discussed in the next paper.

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