# Octonic second-order equations of relativistic quantum mechanics

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We demonstrate a generalization of relativistic quantum mechanics using eight-component value "octons" that generate an associative noncommutative spatial algebra. It is shown that the octonic second-order equation for the eight-component octonic wave function, obtained from the Einstein relation for energy and momentum, describes particles with spin 1/2. It is established that the octonic wave function of a particle in the state with defined spin projection has a specific spatial structure that takes the form of an octonic oscillator with two spatial polarizations: longitudinal linear and transverse circular. © *2009 American Institute of Physics.* [DOI: 10.1063/1.3058644]

## I. INTRODUCTION

Multicomponent hypernumbers[1–5] are widely used in relativistic mechanics, electrodynamics, quantum mechanics, and quantum field theory.[6–24] For instance, the structure of four-component quaternions adequately conforms to the four-dimensional character of space-time and enables the development quaternionic generalizations of quantum mechanics,[6–12] where particles are described by four-component wave functions consisting of scalar and vector parts. However, quaternionic wave functions have no pseudoscalar and pseudovector properties and are transformed incorrectly with respect to spatial inversion. Therefore it seems that eight-component values including scalar, pseudoscalar, vector, and pseudovector components are more appropriate for the description of relativistic quantum systems.

However, attempts to describe relativistic particles by means of different eight-component hypernumbers such as biquaternions,[5,15,16] octonions,[17–21] and multivectors generating associative Clifford algebras[22–24] have not made appreciable progress. For example, the few attempts to describe relativistic particles by means of octonion wave functions are confronted by difficulties connected with octonions nonassociativity.[19] Moreover all systems of hypercomplex numbers, which have been applied up to now for the generalization of quantum mechanics (quaternions, biquaternions, octonions, and multivectors), are the objects of hypercomplex space and do not have any consistent space-geometric interpretation.

Recently we proposed eight-component value "octons"[25] generating a closed noncommutative associative algebra and having a clear well-defined geometric interpretation. From the geometrical point of view an octon is the object of the real three-dimensional space. It is the sum of a scalar, pseudoscalar, vector, and pseudovector. In Ref. 25 octons were successfully applied to the description of the classical electromagnetic field. In this paper we propose the generalization of relativistic quantum mechanics using octonic equations for eight-component octonic wave functions.

## II. ALGEBRA OF OCTONS

To begin with we will briefly review the basic properties of octons. The eight-component octon $\breve{G}$ is defined in the form

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TABLE I. The rules of multiplication and commutation for the octon's unit vectors.

| | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{a}_0$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ |
|---|---|---|---|---|---|---|---|
| $\mathbf{e}_1$ | 1 | $i\mathbf{e}_3$ | $-i\mathbf{e}_2$ | $\mathbf{a}_1$ | $\mathbf{a}_0$ | $i\mathbf{a}_3$ | $-i\mathbf{a}_2$ |
| $\mathbf{e}_2$ | $-i\mathbf{e}_3$ | 1 | $i\mathbf{e}_1$ | $\mathbf{a}_2$ | $-i\mathbf{a}_3$ | $\mathbf{a}_0$ | $i\mathbf{a}_1$ |
| $\mathbf{e}_3$ | $i\mathbf{e}_2$ | $-i\mathbf{e}_1$ | 1 | $\mathbf{a}_3$ | $i\mathbf{a}_2$ | $-i\mathbf{a}_1$ | $\mathbf{a}_0$ |
| $\mathbf{a}_0$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | 1 | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ |
| $\mathbf{a}_1$ | $\mathbf{a}_0$ | $i\mathbf{a}_3$ | $-i\mathbf{a}_2$ | $\mathbf{e}_1$ | 1 | $i\mathbf{e}_3$ | $-i\mathbf{e}_2$ |
| $\mathbf{a}_2$ | $-i\mathbf{a}_3$ | $\mathbf{a}_0$ | $i\mathbf{a}_1$ | $\mathbf{e}_2$ | $-i\mathbf{e}_3$ | 1 | $i\mathbf{e}_1$ |
| $\mathbf{a}_3$ | $i\mathbf{a}_2$ | $-i\mathbf{a}_1$ | $\mathbf{a}_0$ | $\mathbf{e}_3$ | $i\mathbf{e}_2$ | $-i\mathbf{e}_1$ | 1 |

$$\breve{G} = c_0\mathbf{e_0} + c_1\mathbf{e_1} + c_2\mathbf{e_2} + c_3\mathbf{e_3} + d_0\mathbf{a_0} + d_1\mathbf{a_1} + d_2\mathbf{a_2} + d_3\mathbf{a_3},$$

where $\mathbf{e_0} \equiv 1$, values $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are axial unit vectors (pseudovectors), $\mathbf{a_0}$ is the pseudoscalar unit, and $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are polar unit vectors. The octon's components $c_n$ and $d_n$ $(n=0,1,2,3)$ are numbers (complex, in general). Thus the octon is the sum of a scalar, pseudovector, pseudoscalar, and vector. The values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are the polar and axial bases of the octon, respectively. The algebra of octons was discussed in detail in Ref. 25. Here we briefly recall the basic commutation and multiplication rules, which are represented in the Table I.

In Table I and below the value $i$ is the imaginary unit.

## III. OCTONIC WAVE FUNCTION AND SPATIAL OPERATORS

Let us consider the wave function of a relativistic particle in the form of an eight-component octon,

$$\breve{\psi} = \varphi_0 + \varphi_1\mathbf{e_1} + \varphi_2\mathbf{e_2} + \varphi_3\mathbf{e_3} + \chi_0\mathbf{a_0} + \chi_1\mathbf{a_1} + \chi_2\mathbf{a_2} + \chi_3\mathbf{a_3}. \quad (1)$$

The components $\varphi_\alpha(\vec{r},t)$ and $\chi_\alpha(\vec{r},t)$ $(\alpha=0,1,2,3)$ are scalar (complex, in general) functions of spatial coordinates and time. The octonic wave function (1) can be also written in the compact form,

$$\breve{\psi} = \varphi_0 + \overleftrightarrow{\varphi} + \tilde{\chi}_0 + \vec{\chi},$$

where the pseudovector part is indicated by a double arrow "↔," the pseudoscalar part by a wave "∼," and the vector part by an arrow "→." The Lorentz transform of the octonic wave function is represented in Appendix A.

The wave function of a free particle should satisfy an equation, which is obtained from the Einstein relation between particle energy and momentum,

$$E^2 - p^2c^2 = m^2c^4,$$

by substituting the classical momentum $\vec{p}$ and energy $E$ by the corresponding quantum-mechanical operators $\hat{\vec{p}}=-i\hbar\vec{\nabla}$ and $\hat{E}=i\hbar\,\partial/\partial t$. This equation has the following form:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\breve{\psi} = -\frac{m^2c^2}{\hbar^2}\breve{\psi}. \quad (2)$$

Here $c$ is the velocity of light, $m$ is the mass of the particle, and $\hbar$ is the Plank constant. In contrast to the scalar Klein–Gordon equation[26] the expression (2) is the octonic equation since it is written for the octonic function. It is clear that each of the $\breve{\psi}$ components satisfies the scalar Klein–Gordon equation.

The operator in the left part of Eq. (2) can be represented as the product of two operators,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\nabla}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\breve{\psi} = -\frac{m^2c^2}{\hbar^2}\breve{\psi}. \tag{3}$$

Here we assume that the octonic wave function $\breve{\psi}$ is twice continuously differentiable, so $[\vec{\nabla}\times\vec{\nabla}]\breve{\psi}=0$. [The operations of vector $[\times]$ and scalar $(\cdot)$ octonic multiplication were considered in Ref. 25.] Equation (3) can be written in the expanded form,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x_1}\mathbf{a_1} - \frac{\partial}{\partial x_2}\mathbf{a_2} - \frac{\partial}{\partial x_3}\mathbf{a_3}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\mathbf{a_1} + \frac{\partial}{\partial x_2}\mathbf{a_2} + \frac{\partial}{\partial x_3}\mathbf{a_3}\right)\breve{\psi}(\vec{r},t) = -\frac{m^2c^2}{\hbar^2}\breve{\psi}(\vec{r},t). \tag{4}$$

Note that unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ in the left part of Eq. (4) transform the spatial structure of the wave function by means of octonic multiplication. In this sense they can be treated as spatial operators $\hat{\mathbf{a}}_\mathbf{1}$, $\hat{\mathbf{a}}_\mathbf{2}$, $\hat{\mathbf{a}}_\mathbf{3}$, which transform the octon of the wave function. For example, the action of the operator $\hat{\mathbf{a}}_\mathbf{1}$ can be represented as octonic multiplication of the unit vector $\mathbf{a_1}$ and octon $\breve{\psi}$,

$$\hat{\mathbf{a}}_\mathbf{1}\breve{\psi} = \mathbf{a_1}\breve{\psi} = \chi_1 + \chi_0\mathbf{e_1} - i\chi_3\mathbf{e_2} + i\chi_2\mathbf{e_3} + \varphi_1\mathbf{a_0} + \varphi_0\mathbf{a_1} - i\varphi_3\mathbf{a_2} + i\varphi_2\mathbf{a_3}. \tag{5}$$

Further we will use symbolic designations $\hat{\mathbf{a}}_\mathbf{1}$, $\hat{\mathbf{a}}_\mathbf{2}$, $\hat{\mathbf{a}}_\mathbf{3}$ in the operator part of equations but $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ designations in wave functions. Then Eq. (4) can be rewritten in the operator form,

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x_1}\hat{\mathbf{a}}_1 - \frac{\partial}{\partial x_2}\hat{\mathbf{a}}_\mathbf{2} - \frac{\partial}{\partial x_3}\hat{\mathbf{a}}_\mathbf{3}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\hat{\mathbf{a}}_\mathbf{1} + \frac{\partial}{\partial x_2}\hat{\mathbf{a}}_\mathbf{2} + \frac{\partial}{\partial x_3}\hat{\mathbf{a}}_\mathbf{3}\right)\breve{\psi}(\vec{r},t) = -\frac{m^2c^2}{\hbar^2}\breve{\psi}(\vec{r},t).$$

On the other hand expression (5) can be represented in equivalent matrix form as the action of the matrix operator $\hat{\mathbf{a}}_\mathbf{1}$ on the eight-component column of the wave function, so $\hat{\mathbf{a}}_\mathbf{1}$ can be written as the $8\times 8$ matrix,

$$\hat{\mathbf{a}}_\mathbf{1}\begin{pmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \chi_0\\ \chi_1\\ \chi_2\\ \chi_3\end{pmatrix} = \begin{pmatrix}\chi_1\\ \chi_0\\ -i\chi_3\\ i\chi_2\\ \varphi_1\\ \varphi_0\\ -i\varphi_3\\ i\varphi_2\end{pmatrix} \Rightarrow \hat{\mathbf{a}}_\mathbf{1} = \begin{pmatrix}0&0&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&-i\\ 0&0&0&0&0&0&i&0\\ 0&1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0\\ 0&0&i&0&0&0&0&0\end{pmatrix}.$$

The matrix representations for the rest octonic operators $\hat{\mathbf{a}}_\mathbf{2}$, $\hat{\mathbf{a}}_\mathbf{3}$, $\hat{\mathbf{a}}_\mathbf{0}$, $\hat{\mathbf{e}}_\mathbf{1}$, $\hat{\mathbf{e}}_\mathbf{2}$, $\hat{\mathbf{e}}_\mathbf{3}$ are given in the Appendix B. Further we will use the vector designation for the operators of the octon's basis since it is more compact.

## IV. EIGENVALUES AND EIGENFUNCTIONS OF OCTONIC SPATIAL OPERATORS

Let us consider the eigenvalues and eigenfunctions of spatial operators used as an example the operator $\hat{\mathbf{e}}_\mathbf{3}$, which will be used below for the description of the particle in a homogeneous magnetic field. The equation for the eigenvalues and eigenfunctions in this case has the following form:

$$\hat{\mathbf{e}}_\mathbf{3}\breve{\psi} = \lambda\breve{\psi}. \tag{6}$$

Performing octonic multiplication in the left part of (6) and equating components, we obtain the system of eight scalar equations, which is equivalent to the following system:

$$\lambda^2 = 1,$$

$$\varphi_3 = \lambda\varphi_0,$$

$$\varphi_2 = i\lambda\varphi_1, \tag{7}$$

$$\chi_3 = \lambda\chi_0,$$

$$\chi_2 = i\lambda\chi_1.$$

The first equation in (7) shows that $\lambda = \pm 1$. For each eigenvalue $\lambda$ there is a four-dimensional subspace of eigenfunctions. Taking into account (7) we can choose the set of functions

$$(1+\mathbf{e_3}),(\mathbf{e_1}+i\mathbf{e_2}),(\mathbf{a_0}+\mathbf{a_3}),(\mathbf{a_1}+i\mathbf{a_2}) \tag{8}$$

as the basis of the subspace corresponding to eigenvalue $\lambda=+1$ and set of functions

$$(1-\mathbf{e_3}),(\mathbf{e_1}-i\mathbf{e_2}),(\mathbf{a_0}-\mathbf{a_3}),(\mathbf{a_1}-i\mathbf{a_2}) \tag{9}$$

as the basis of the subspace corresponding to $\lambda=-1$. Then arbitrary eigenfunctions of the operator $\hat{\mathbf{e}}_3$ corresponding to $\lambda=\pm 1$ can be represented in the form of linear combinations of the basis functions (8) or (9),

$$\breve{\psi}_{\lambda=1} = F_1^{(1)}(\vec{r},t)(1+\mathbf{e_3}) + F_2^{(1)}(\vec{r},t)(\mathbf{e_1}+i\mathbf{e_2}) + F_3^{(1)}(\vec{r},t)(\mathbf{a_0}+\mathbf{a_3}) + F_4^{(1)}(\vec{r},t)(\mathbf{a_1}+i\mathbf{a_2}), \tag{10}$$

$$\breve{\psi}_{\lambda=-1} = F_1^{(-1)}(\vec{r},t)(1-\mathbf{e_3}) + F_2^{(-1)}(\vec{r},t)(\mathbf{e_1}-i\mathbf{e_2}) + F_3^{(-1)}(\vec{r},t)(\mathbf{a_0}-\mathbf{a_3}) + F_4^{(-1)}(\vec{r},t)(\mathbf{a_1}-i\mathbf{a_2}), \tag{11}$$

where $F_\alpha^{(\lambda)}(\vec{r},t)$ are arbitrary scalar functions of space coordinates and time ($\alpha=1,2,3,4$; $\lambda = \pm 1$).

It is obvious that eigenvalues and eigenfunctions of the operator $\hat{\mathbf{e}}_3$ could also be obtained using the matrix representation.

The eigenfunctions of other octonic spatial operators are shown in Appendix C.

## V. OCTONIC EQUATION FOR A RELATIVISTIC PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

To describe a particle in an external electromagnetic field the following change in quantum-mechanical operators in equations should be made:[26]

$$\hat{E} \to \hat{E} - e\Phi, \quad \hat{\vec{p}} \to \hat{\vec{p}} - \frac{e}{c}\vec{A}, \tag{12}$$

where $\Phi$ and $\vec{A}$ are scalar and vector potentials of the electromagnetic field and $e$ is the particle charge ($e<0$ for the electron). The change (12) is equivalent to the following change in differential operators:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \frac{ie}{\hbar}\Phi, \quad \vec{\nabla} \to \vec{\nabla} - \frac{ie}{\hbar c}\vec{A}. \tag{13}$$

Using substitution (13), Eq. (3) can be written as

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi - \vec{\nabla} + \frac{ie}{\hbar c}\vec{A}\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \frac{ie}{\hbar c}\Phi + \vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)\breve{\psi} = -\frac{m^2c^2}{\hbar^2}\breve{\psi}. \tag{14}$$

The multiplication of octonic operators in the left part of (14) leads to the following equation:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{2ie}{\hbar c}\left((\vec{A}\cdot\vec{\nabla}) + \frac{\Phi}{c}\frac{\partial}{\partial t}\right) + \frac{m^2c^2}{\hbar^2} + \frac{e^2}{\hbar^2 c^2}(A^2 - \Phi^2)\right]\breve{\psi} - \frac{e}{\hbar c}\overleftrightarrow{H}\breve{\psi} + \frac{ie}{\hbar c}\vec{E}\breve{\psi} = 0. \quad (15)$$

Here we have taken into account that $\vec{E}=-\vec{\nabla}\Phi-(1/c)(\partial\vec{A}/\partial t)$ is the vector of the electric field and $\overleftrightarrow{H}=-i[\vec{\nabla}\times\vec{A}]$ is the pseudovector of the magnetic field. $(\vec{\nabla}\cdot\vec{A})+(1/c)(\partial\Phi/\partial t)=0$ in the Lorentz gauge. The scalar $(\vec{\nabla}\cdot\vec{A})$ and vector $[\vec{\nabla}\times\vec{A}]$ octonic products were considered in Ref. 25. Note that the octonic equation (15) encloses the specific terms $(e/\hbar c)\overleftrightarrow{H}\breve{\psi}$ and $(ie/\hbar c)\vec{E}\breve{\psi}$, where the fields $\vec{E}$ and $\overleftrightarrow{H}$ play the role of spatial octonic operators.

## VI. OCTONIC EQUATION FOR A NONRELATIVISTIC PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

The nonrelativistic octonic equation for a particle in an electromagnetic field can be obtained directly from the Schrödinger equation. The octonic Hamiltonian for the free particle has the form

$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m} = -\frac{\hbar^2}{2m}\Delta. \quad (16)$$

The octonic equation for the octonic wave function corresponding to Hamiltonian (16) can be written in the following form:

$$i\hbar\frac{\partial\breve{\psi}}{\partial t} = \hat{H}_0\breve{\psi}.$$

In the absence of an electromagnetic field this equation is equivalent to eight scalar Schrödinger equations (one for each component of the octonic wave function).

To describe a particle in an electromagnetic field the quantum-mechanical operators should be changed as follows: $\hat{H}_0 \to \hat{H}=\hat{H}_0+e\Phi$, $\vec{\nabla}\to\vec{\nabla}-(ie/\hbar c)\vec{A}$. In this case we obtain

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)^2 + e\Phi. \quad (17)$$

Multiplying the octonic operators in (17) we get

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + \frac{i\hbar e}{2mc}(\vec{\nabla}\cdot\vec{A}) + \frac{i\hbar e}{mc}(\vec{A}\cdot\vec{\nabla}) + \frac{e^2}{2mc^2}A^2 + e\Phi - \frac{\hbar e}{2mc}\overleftrightarrow{H}. \quad (18)$$

Thus the nonstationary octonic equation for a nonrelativistic particle in an electromagnetic field can be written as

$$i\hbar\frac{\partial\breve{\psi}}{\partial t} = \hat{H}\breve{\psi}.$$

In a stationary state with energy $E$ the wave function satisfies the following equation:

$$-\frac{\hbar^2}{2m}\Delta\breve{\psi} + \frac{i\hbar e}{2mc}(\vec{\nabla}\cdot\vec{A})\breve{\psi} + \frac{i\hbar e}{mc}(\vec{A}\cdot\vec{\nabla})\breve{\psi} + \frac{e^2}{2mc^2}A^2\breve{\psi} + e\Phi\breve{\psi} - \frac{\hbar e}{2mc}\overleftrightarrow{H}\breve{\psi} = E\breve{\psi}. \quad (19)$$

Note that the last term $-(\hbar e/2mc)\overleftrightarrow{H}$ in the Hamiltonian (18) is a pseudovector operator, which transforms the spatial structure of the octonic wave function.

## VII. NONRELATIVISTIC PARTICLE IN HOMOGENEOUS MAGNETIC FIELD

Let us find the stationary state of a particle in a homogeneous magnetic field. Let the pseudovector of the magnetic field intensity be directed along the $Z$ axis in the $XYZ$ basis,

$$\overleftrightarrow{H} = B\mathbf{e_3}.$$

We select the vector potential in the gauge $(\vec{\nabla}\cdot\vec{A})=0$,

$$\vec{A} = A_y\mathbf{a_2} = Bx\mathbf{a_2}.$$

Then the Eq. (19) can be written in the following form:

$$-\frac{\hbar^2}{2m}\Delta\breve{\psi} + \frac{i\hbar e}{mc}Bx\frac{\partial\breve{\psi}}{\partial y} + \frac{e^2}{2mc^2}B^2x^2\breve{\psi} - \frac{\hbar e}{2mc}B\hat{\mathbf{e}}_\mathbf{3}\breve{\psi} = E\breve{\psi}. \tag{20}$$

Note that operators $\hat{p}_y=-i\hbar\,\partial/\partial y$ and $\hat{p}_z=-i\hbar\,\partial/\partial z$ commute with Hamiltonian (18), and all of them have common eigenfunctions. Therefore we will search for a solution of Eq. (20) in the following form:

$$\breve{\psi}(\vec{r}) = \breve{W}(x)e^{(i/\hbar)(p_y y+p_z z)}, \tag{21}$$

where $p_y$ and $p_z$ are the motion integrals. Substituting (21) into (20), we get

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} - \frac{ep_y}{mc}Bx + \frac{e^2}{2mc^2}B^2x^2 - \frac{\hbar e}{2mc}B\hat{\mathbf{e}}_\mathbf{3}\right]\breve{W} = E\breve{W}. \tag{22}$$

Note that Hamiltonian (18) commutes also with the octonic pseudovector operator $\hat{\mathbf{e}}_\mathbf{3}$, so the solutions of Eq. (20) are also eigenfunctions of the operator $\hat{\mathbf{e}}_\mathbf{3}$. This means that in this task there is another quantum number $\lambda$ (eigenvalue of the operator $\hat{\mathbf{e}}_\mathbf{3}$), which takes on the values $\lambda=\pm 1$. Therefore we will search for a solution of Eq. (22) in the form of eigenfunctions of the operator $\hat{\mathbf{e}}_\mathbf{3}$ [see (10) and (11)],

$$\breve{W} = F_1^{(\lambda)}(x)(1+\lambda\mathbf{e_3}) + F_2^{(\lambda)}(x)(\mathbf{e_1}+\lambda i\mathbf{e_2}) + F_3^{(\lambda)}(x)(\mathbf{a_0}+\lambda\mathbf{a_3}) + F_4^{(\lambda)}(x)(\mathbf{a_1}+\lambda i\mathbf{a_2}). \tag{23}$$

Then the operator in the left part of (22) is scalar, and the equation can be written as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\breve{W}}{\partial x^2} + \frac{p_y^2}{2m}\breve{W} + \frac{p_z^2}{2m}\breve{W} - \frac{ep_y}{mc}Bx\breve{W} + \frac{e^2}{2mc^2}B^2x^2\breve{W} - \frac{\hbar e}{2mc}B\lambda\breve{W} = E\breve{W}. \tag{24}$$

In fact, Eq. (24) is the system of four identical scalar equations for the functions $F_\alpha^{(\lambda)}(x)$, where $\alpha=1,2,3,4$ [see (23)]. For the fixed $\lambda$ the functions $F_\alpha^{(\lambda)}(x)$ can differ only by a constant.

The equation for each function $F_\alpha^{(\lambda)}(x)$ has the following form:

$$-\frac{\hbar^2}{2m}\frac{\partial^2F_\alpha^{(\lambda)}}{\partial x^2} + \frac{p_y^2}{2m}F_\alpha^{(\lambda)} + \frac{p_z^2}{2m}F_\alpha^{(\lambda)} - \frac{ep_y}{mc}BxF_\alpha^{(\lambda)} + \frac{e^2}{2mc^2}B^2x^2F_\alpha^{(\lambda)} - \frac{\hbar e}{2mc}B\lambda F_\alpha^{(\lambda)} = EF_\alpha^{(\lambda)}. \tag{25}$$

After algebraic transformations Eq. (25) can be represented in the form

$$\frac{\partial^2F_\alpha^{(\lambda)}}{\partial x^2} + \frac{2m}{\hbar^2}\left\{\left(E - \frac{p_z^2}{2m} + \lambda\frac{\hbar e}{2mc}B\right) - \frac{m}{2}\left(\frac{eB}{mc}\right)^2\left(x - \frac{cp_y}{eB}\right)^2\right\}F_\alpha^{(\lambda)} = 0. \tag{26}$$

On the basis of Eq. (26) we obtain the expression for the energy spectrum of a nonrelativistic particle in a homogeneous magnetic field,

$$E_{n,\lambda} = \frac{p_z^2}{2m} + \frac{\hbar|e|B}{mc}\left(n+\frac{1}{2}\right) - \lambda\frac{\hbar e}{2mc}B. \tag{27}$$

This set of energies is absolutely identical to the energy spectrum obtained from the nonrelativistic Pauli equation.[27] Thus the octonic equation (19) enables the description of the spin interaction with the magnetic field.

## VIII. RELATIVISTIC PARTICLE IN A HOMOGENEOUS MAGNETIC FIELD

Let us consider a relativistic particle in an external magnetic field directed along the $Z$ axis, $\overleftrightarrow{H}=B\mathbf{e_3}$, $\vec{A}=A_y\mathbf{a_2}=Bx\mathbf{a_2}$, and assume the condition $(\vec{\nabla}\cdot\vec{A})=0$. Then the octonic equation for a relativistic particle (15) in the $XYZ$ basis can be written as

$$\frac{1}{c^2}\frac{\partial^2\breve{\psi}}{\partial t^2}-\Delta\breve{\psi}+\frac{2ie}{\hbar c}Bx\frac{\partial\breve{\psi}}{\partial y}+\frac{m^2c^2}{\hbar^2}\breve{\psi}+\frac{e^2}{\hbar^2c^2}B^2x^2\breve{\psi}-\frac{e}{\hbar c}\overleftrightarrow{H}\breve{\psi}=0.$$

For the stationary state with the energy $E$ we get

$$\left[-\Delta+\frac{2ie}{\hbar c}Bx\frac{\partial}{\partial y}+\frac{m^2c^2}{\hbar^2}+\frac{e^2}{\hbar^2c^2}B^2x^2-\frac{e}{\hbar c}B\hat{\mathbf{e}}_\mathbf{3}\right]\breve{\psi}=\frac{E^2}{\hbar^2c^2}\breve{\psi}. \tag{28}$$

This equation can be considered as the equation for the eigenvalues and eigenfunctions of the complicated operator placed in the left part. Since this operator commutes with operators $\hat{p}_y$ and $\hat{p}_z$, all of them have the general system of eigenfunctions. Therefore we will search for a solution of (28) in the form

$$\breve{\psi}=\breve{W}(x)e^{(i/\hbar)(p_yy+p_zz)}, \tag{29}$$

where $p_y$ and $p_z$ are the motion integrals. Substituting (29) into (28) we get

$$\left[\frac{p_y^2}{\hbar^2}+\frac{p_z^2}{\hbar^2}-\frac{\partial^2}{\partial x^2}-\frac{2ep_y}{\hbar^2c}Bx+\frac{m^2c^2}{\hbar^2}+\frac{e^2}{\hbar^2c^2}B^2x^2-\frac{e}{\hbar c}B\hat{\mathbf{e}}_\mathbf{3}\right]\breve{W}=\frac{E^2}{\hbar^2c^2}\breve{W}. \tag{30}$$

Note that operator in the left part of (30) commutes also with $\hat{\mathbf{e}}_\mathbf{3}$, so we can search for a solution as the eigenfunctions of the operator $\hat{\mathbf{e}}_\mathbf{3}$ (10) and (11),

$$\breve{W}=F_1^{(\lambda)}(x)(1+\lambda\mathbf{e_3})+F_2^{(\lambda)}(x)(\mathbf{e_1}+\lambda i\mathbf{e_2})+F_3^{(\lambda)}(x)(\mathbf{a_0}+\lambda\mathbf{a_3})+F_4^{(\lambda)}(x)(\mathbf{a}_1+\lambda i\mathbf{a}_2).$$

So we get scalar equations for the functions $F_\alpha^{(\lambda)}(x)$ (where $\alpha=1,2,3,4$),

$$\frac{p_y^2}{\hbar^2}F_\alpha^{(\lambda)}+\frac{p_z^2}{\hbar^2}F_\alpha^{(\lambda)}-\frac{\partial^2F_\alpha^{(\lambda)}}{\partial x^2}-\frac{2ep_y}{\hbar^2c}BxF_\alpha^{(\lambda)}+\frac{m^2c^2}{\hbar^2}F_\alpha^{(\lambda)}+\frac{e^2}{\hbar^2c^2}B^2x^2F_\alpha^{(\lambda)}-\frac{e}{\hbar c}B\lambda F_\alpha^{(\lambda)}=\frac{E^2}{\hbar^2c^2}F_\alpha^{(\lambda)}. \tag{31}$$

After some algebraic transformations Eq. (31) can be represented as

$$\frac{\partial^2F_\alpha^{(\lambda)}}{\partial x^2}+\left[\left(\frac{E^2}{\hbar^2c^2}-\frac{p_z^2}{\hbar^2}-\frac{m^2c^2}{\hbar^2}+\lambda\frac{eB}{\hbar c}\right)-\left(\frac{eB}{\hbar c}\right)^2\left(x-\frac{cp_y}{eB}\right)^2\right]F_\alpha^{(\lambda)}=0.$$

Then the energy spectrum is defined by the following expression:

$$E_{n,\lambda}^2=m^2c^4+p_z^2c^2+|e|B\hbar c(2n+1)-\lambda eB\hbar c. \tag{32}$$

This set of energies is absolutely identical to the energy spectrum obtained from the relativistic second-order equation following from the Dirac equation.[26] Note that in the nonrelativistic limit the energy levels (32) are transformed to (27) for the levels counted out of the rest energy.

If the wave function is the eigenfunction of the operator $\hat{\mathbf{e}}_\mathbf{3}$, then some general statements about the spatial structure of the wave function can be made. In the stationary state with energy $E$ the wave function can be represented in the following form:

$$\breve{\psi}(\vec{r},t)=\breve{\psi}(\vec{r})e^{-i\omega t}, \tag{33}$$

where $\omega=E/\hbar$. If the eigenvalue of the operator $\hat{\mathbf{e}}_\mathbf{3}$ is defined, the spatial part of the wave function can be written as the linear combination,

$$\breve{\psi}(\vec{r}) = F_1(\vec{r})(1 + \lambda \mathbf{e_3}) + F_2(\vec{r})(\mathbf{e_1} + \lambda i \mathbf{e_2}) + F_3(\vec{r})(\mathbf{a_0} + \lambda \mathbf{a_3}) + F_4(\vec{r})(\mathbf{a_1} + \lambda i \mathbf{a_2}). \tag{34}$$

The wave function (33) with spatial part (34) has a simple geometrical image. The real and imaginary parts of the component $(1+\lambda \mathbf{e_3})e^{-i\omega t}$ are a combination of a pseudovector directed parallel to the $Z$ axis and a scalar oscillating with the frequency $\omega$. Here the phase difference between oscillations of scalar and pseudovector parts equals 0 in case of $\lambda=1$ or $\pi$ in case of $\lambda=-1$. The real and imaginary parts of the component $(\mathbf{a_1}+\lambda i\mathbf{a_2})e^{-i\omega t}$ have the form of polar vectors rotating in the plane perpendicular to the $Z$ axis also with the frequency $\omega$. The direction of the rotation depends on the sign of $\lambda$. When $\lambda=+1$ a vector of angular velocity is directed along the $Z$ axis but when $\lambda=-1$ this vector has the opposite direction. The rest of the components of the wave function (34) have a similar interpretation.

Thus in octonic quantum mechanics the wave function of the particle with definite spin projection has the space-time structure of the octonic oscillator (33) and (34) with longitudinal linear and transverse circular spatial polarizations.

## IX. DISCUSSION

One of the starting points for constructing the relativistic quantum theory is the Einstein relation between energy and momentum of the particle, which is written as

$$E^2 = \vec{p}^2 c^2 + m^2 c^4.$$

In relativistic quantum mechanics this relation is written as the operator equation for the wave function

$$(\hat{E}^2 - \hat{\vec{p}}^2 c^2)\psi = m^2 c^4 \psi. \tag{35}$$

This equation contains the scalar operator $\hat{\vec{p}}^2=\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2$. The form of the operator $\hat{\vec{p}}$, in general, is ambiguous. Equation (35) can be realized by means of different operators and wave functions. For example, choosing a scalar wave function and the operator $\hat{\vec{p}}$ represented by Gibbs unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$,

$$\hat{\vec{p}} = \hat{p}_x \vec{i} + \hat{p}_y \vec{j} + \hat{p}_z \vec{k},$$

we can obtain the well known Klein–Gordon equation describing spinless particles. Moreover one can consider the following matrix expression for the $\hat{\vec{p}}$ operator:

$$\hat{\vec{p}} = \hat{p}_x \hat{\sigma}_x + \hat{p}_y \hat{\sigma}_y + \hat{p}_z \hat{\sigma}_z,$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$ are some matrices. For example, the Pauli matrices or the Dirac matrices can be used as $\hat{\sigma}_s$ $(s=\{x,y,z\})$. It is only necessary that these matrices anticommute and that $\sigma_s^2=1$ to obtain $\hat{\vec{p}}^2=\hat{p}_x^2+\hat{p}_y^2+\hat{p}_z^2$. There are also other approaches to the operator $\hat{\vec{p}}$ representation by quaternions.[14,18]

From this point of view it is not surprising that the proposed octonic wave function and octonic momentum operator in the form

$$\hat{\vec{p}} = \hat{p}_x \mathbf{a_1} + \hat{p}_y \mathbf{a_2} + \hat{p}_z \mathbf{a_3} \tag{36}$$

also realize the right representation of Eq. (35). However, we emphasize that octonic representation of the operator $\hat{\vec{p}}$ (36) is distinguished since it allows the conservation of the vector interpretation of $\hat{\vec{p}}$ and ensures the correct space transformation properties.

In octonic algebra the classical Einstein relation for the particle in an electromagnetic field can be represented in the form

$$(E - e\Phi + \vec{p}c - e\vec{A})(E - e\Phi - \vec{p}A + e\vec{A}) = m^2c^4. \tag{37}$$

Performing octonic multiplication in the left part of (37) and gathering terms, we get

$$E^2 - 2e\Phi E + e^2\Phi^2 - \vec{p}^2c^2 + 2ec(\vec{p}\cdot\vec{A}) - e^2\vec{A}^2 = m^2c^4$$

or

$$(E - e\Phi)^2 - \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 c^2 = m^2c^4. \tag{38}$$

Thus octonic relations (37) and (38) in classical relativistic mechanics are absolutely equivalent. Note that these relations are as valid in octon's algebra as in Gibbs vectors algebra.

However, in quantum mechanics the situation is cardinally changed. The octonic operator equation obtained from (37) can be represented as

$$(\hat{E} - e\Phi + \hat{\vec{p}}c - e\vec{A})(\hat{E} - e\Phi - \hat{\vec{p}}c + e\vec{A})\breve{\psi} = m^2c^4\breve{\psi}, \tag{39}$$

but the equation obtained from (38) can be written as

$$\left((\hat{E} - e\Phi)^2 - \left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2 c^2\right)\breve{\psi} = m^2c^4\breve{\psi}. \tag{40}$$

Equations (39) and (40) are essentially nonequivalent. In fact, Eq. (39) after octonic multiplication leads us to (15) but Eq. (40) leads us to the following one:

$$\left[-\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{2ie}{\hbar c}\left((\vec{A}\cdot\vec{\nabla}) + \frac{\Phi}{c}\frac{\partial}{\partial t}\right) + \frac{m^2c^2}{\hbar^2} + \frac{e^2}{\hbar^2c^2}(A^2 - \Phi^2)\right]\breve{\psi} - \frac{e}{\hbar c}\overleftrightarrow{H}\breve{\psi} = 0.$$

This expression differs from (15) by the absence of the term describing the interaction of the particle with the electric field. Therefore the representation of the Einstein relation in the form (37) is more adequate than (38) from the point of view of the description of the particle in the electromagnetic field.

Consequently in octonic quantum mechanics the second-order equation correctly describing the interaction of spin with the electromagnetic field is obtained directly from the Einstein relation in contrast to the Dirac theory where the first-order equation is used to obtain this equation.

Also the algebra of octons enables the derivation of the nonrelativistic equation, which correctly describes interaction of spin with a magnetic field, directly from the Schrödinger equation. Indeed the Schrödinger Hamiltonian contains the operator $\hat{\vec{p}}^2$, which is modified in the presence of magnetic field by

$$\hat{\vec{p}}^2 \to \left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2. \tag{41}$$

The octonic multiplication of operators in (41) leads to the appearance of the term, which corresponds to the interaction of spin with the magnetic field [see (18)]. In the Pauli theory the equation analogous to (19) is postulated. Accordingly the application of the octon's algebra leads to the adequate description of relativistic and nonrelativistic quantum particles in an external electromagnetic field. In octonic quantum mechanics a concept of spin is closely connected with spatial properties of octonic operators and octonic wave functions. In the state with definite spin projection the wave function has a complicated space-time structure in which vector and pseudovector components perform either spatial oscillations along the $Z$ direction in the case of longitudinal polarization or rotations in the perpendicular plane in the case of transverse polarization of the octonic oscillator.

## X. CONCLUSION

In this paper we proposed a scheme for constructing relativistic quantum mechanics using octonic spatial operators and octonic wave functions. It was shown that the octonic second-order equation, corresponding to the Einstein relation between energy and momentum, correctly describes the interaction between spin and the electromagnetic field. It is established that the octonic wave function of a particle in the state with defined spin projection has the specific space-time structure in the form of an octonic oscillator with two spatial polarizations: longitudinal linear and transverse circular.

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## APPENDIX A: LORENTZ TRANSFORMATION OF THE OCTONIC WAVE FUNCTION

The Lorentz transformation of the octonic wave function $\breve{\psi}$ is given by the octon

$$\breve{S} = \text{ch}(u/2) - \text{sh}(u/2)\vec{n}, \tag{A1}$$

where th $u = v/c$, $v$ is the velocity of motion along the unit vector $\vec{n}$. The transformed function $\breve{\psi}'$ is defined by the octonic product

$$\breve{\psi}' = \breve{S}\breve{\psi}\breve{S}. \tag{A2}$$

The transformed function $\breve{\psi}'$ can be written on the basis of (A1) and (A2) as

$$\begin{aligned}\breve{\psi}' &= [\text{ch}(u/2) - \text{sh}(u/2)\vec{n}](\varphi_0 + \vec{\tilde{\varphi}} + \tilde{\chi}_0 + \vec{\chi})[\text{ch}(u/2) - \text{sh}(u/2)\vec{n}] \\ &= \varphi_0 \text{ ch } u + \vec{\chi} - \varphi_0\vec{n} \text{ sh } u - (\vec{n}\cdot\vec{\chi})\text{sh } u - (1 - \text{ch } u)(\vec{n}\cdot\vec{\chi})\vec{n} + \tilde{\chi}_0 \text{ ch } u + \vec{\tilde{\varphi}} - \tilde{\chi}_0\vec{n} \text{ sh } u \\ &\quad - (\vec{n}\cdot\vec{\tilde{\varphi}})\text{sh } u - (1 - \text{ch } u)(\vec{n}\cdot\vec{\tilde{\varphi}})\vec{n}.\end{aligned}$$

The components of transformed octonic function have the following structure:

$$\varphi_0' = \varphi_0 \text{ ch } u - (\vec{n}\cdot\vec{\chi})\text{sh } u,$$

$$\tilde{\chi}_0' = \tilde{\chi}_0 \text{ ch } u - (\vec{n}\cdot\vec{\tilde{\varphi}})\text{sh } u,$$

$$\vec{\chi}' = \vec{\chi} - \varphi_0\vec{n} \text{ sh } u - (1 - \text{ch } u)(\vec{n}\cdot\vec{\chi})\vec{n},$$

$$\vec{\tilde{\varphi}}' = \vec{\tilde{\varphi}} - \tilde{\chi}_0\vec{n} \text{ sh } u - (1 - \text{ch } u)(\vec{n}\cdot\vec{\tilde{\varphi}})\vec{n}.$$

It is seen that scalar $\varphi_0$ and pseudoscalar $\tilde{\chi}_0$ parts of the octonic wave function are transformed as the timelike components. The longitudinal (relative to the direction $\vec{n}$) components of the octonic wave function $\vec{\tilde{\varphi}}_\parallel$ and $\vec{\chi}_\parallel$ are transformed as coordinates while transverse components $\vec{\tilde{\varphi}}_\perp$ and $\vec{\chi}_\perp$ are not transformed.

## APPENDIX B: MATRIX FORM FOR OPERATORS OF OCTONIC BASIS

The operators of octonic basis can be represented in the following matrix form [see (10)]:

$$\hat{\mathbf{a}}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\mathbf{a}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{a}}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{e}}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad \hat{\mathbf{e}}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{e}}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

TABLE II. The simplest eigenfunctions of spatial octonic operators.

| | λ=+1 | | | | λ=−1 | | | |
|---|---|---|---|---|---|---|---|---|
| $\hat{\mathbf{e}}_1$ | $(1+\mathbf{e}_1)$ | $(\mathbf{e}_2+i\mathbf{e}_3)$ | $(\mathbf{a}_0+\mathbf{a}_1)$ | $(\mathbf{a}_2+i\mathbf{a}_3)$ | $(1-\mathbf{e}_1)$ | $(\mathbf{e}_2-i\mathbf{e}_3)$ | $(\mathbf{a}_0-\mathbf{a}_1)$ | $(\mathbf{a}_2-i\mathbf{a}_3)$ |
| $\hat{\mathbf{e}}_2$ | $(1+\mathbf{e}_2)$ | $(\mathbf{e}_3+i\mathbf{e}_1)$ | $(\mathbf{a}_0+\mathbf{a}_2)$ | $(\mathbf{a}_3+i\mathbf{a}_1)$ | $(1-\mathbf{e}_2)$ | $(\mathbf{e}_3-i\mathbf{e}_1)$ | $(\mathbf{e}_1+i\mathbf{e}_2)$ | $(\mathbf{a}_3-i\mathbf{a}_1)$ |
| $\hat{\mathbf{e}}_3$ | $(1+\mathbf{e}_3)$ | $(\mathbf{e}_1+i\mathbf{e}_2)$ | $(\mathbf{a}_0+\mathbf{a}_3)$ | $(\mathbf{a}_1+i\mathbf{a}_2)$ | $(1-\mathbf{e}_3)$ | $(\mathbf{e}_1-i\mathbf{e}_2)$ | $(\mathbf{a}_0-\mathbf{a}_3)$ | $(\mathbf{a}_1-i\mathbf{a}_2)$ |
| $\hat{\mathbf{a}}_0$ | $(1+\mathbf{a}_0)$ | $(\mathbf{a}_1+\mathbf{e}_1)$ | $(\mathbf{a}_2+\mathbf{e}_2)$ | $(\mathbf{a}_3+\mathbf{e}_3)$ | $(1-\mathbf{a}_0)$ | $(\mathbf{a}_1-\mathbf{e}_1)$ | $(\mathbf{a}_2-\mathbf{e}_2)$ | $(\mathbf{a}_3-\mathbf{e}_3)$ |
| $\hat{\mathbf{a}}_1$ | $(1+\mathbf{a}_1)$ | $(\mathbf{a}_2+i\mathbf{e}_3)$ | $(\mathbf{a}_0+\mathbf{e}_1)$ | $(\mathbf{e}_2+i\mathbf{a}_3)$ | $(1-\mathbf{a}_1)$ | $(\mathbf{a}_2-i\mathbf{e}_3)$ | $(\mathbf{a}_0-\mathbf{e}_1)$ | $(\mathbf{e}_2-i\mathbf{a}_3)$ |
| $\hat{\mathbf{a}}_2$ | $(1+\mathbf{a}_2)$ | $(\mathbf{a}_3+i\mathbf{e}_1)$ | $(\mathbf{a}_0+\mathbf{e}_2)$ | $(\mathbf{e}_3+i\mathbf{a}_1)$ | $(1-\mathbf{a}_2)$ | $(\mathbf{a}_3-i\mathbf{e}_1)$ | $(\mathbf{a}_0-\mathbf{e}_2)$ | $(\mathbf{e}_3-i\mathbf{a}_1)$ |
| $\hat{\mathbf{a}}_3$ | $(1+\mathbf{a}_3)$ | $(\mathbf{a}_1+i\mathbf{e}_2)$ | $(\mathbf{a}_0+\mathbf{e}_3)$ | $(\mathbf{e}_1+i\mathbf{a}_2)$ | $(1-\mathbf{a}_3)$ | $(\mathbf{a}_1-i\mathbf{e}_2)$ | $(\mathbf{a}_0-\mathbf{e}_3)$ | $(\mathbf{e}_1-i\mathbf{a}_2)$ |

$$\hat{\mathbf{a}}_0=\begin{pmatrix}0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\end{pmatrix}.$$

## APPENDIX C: THE SIMPLEST EIGENFUNCTIONS OF OCTONIC OPERATORS

All operators of octonic basis have two eigenvalues $\lambda=\pm 1$, which are fourthly degenerate. The simplest eigenfunctions are represented in the Table II.


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